ASYMPTOTIC THEORY OF CERTAIN "GOODNESS OF FIT" CRITERIA BASED ON STOCHASTIC PROCESSES

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1. Summary. The statistical problem treated is that of testing the hypothesis that n independent, identically distributed random variables have a specified continuous distribution function $F(x)$. If $F_n(x)$ is the empirical cumulative distribution function and $\psi(t)$ is some nonnegative weight function $(0 \leqq t \leqq 1)$, we consider $n^{\frac{1}{2}} \sup_{-\infty<x<\infty} \{| F(x) - F_n(x) | \psi^{\frac{1}{2}}[F(x)]\}$ and $n\int_{-\infty}^{\infty} [F(x) - F_n(x)]^2 \psi[F(x)]\, dF(x)$. A general method for calculating the limiting distributions of these criteria is developed by reducing them to corresponding problems in stochastic processes, which in turn lead to more or less classical eigenvalue and boundary value problems for special classes of differential equations. For certain weight functions including $\psi = 1$ and $\psi = 1/[t(1 - t)]$ we give explicit limiting distributions. A table of the asymptotic distribution of the von Mises ω^2 criterion is given.

2. Introduction. One method of testing the hypothesis that n observations have been drawn from a population with specified distribution function $F(x)$ is to compare the empirical histogram based on dividing the real line into intervals with the hypothetical histogram by means of the χ^2 tests. A test which does not involve a subjective grouping of the data consists of comparing the empirical cumulative distribution function with the hypothetical distribution function. Let $F_n(x)$ be the empirical distribution function based on n observations; that is, $F_n(x) = k/n$ if k observations are $\leqq x$ for $k = 0, 1, \cdots, n$. We wish to consider a convenient measure of the discrepancy or "distance" between two distribution functions. (For a more detailed discussion cf. Wald and Wolfowitz [21].) In accordance with the usual notions of a metric in function space, we treat the following measures:

$$n \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \psi[F(x)] dF = W_n^2, \tag{2.1}$$

$$\sup_{-\infty<x<\infty} \sqrt{n}\, | F_n(x) - F(x) | \sqrt{\psi[F(x)]} = K_n, \tag{2.2}$$

where $\psi(t)$ $(\geqq 0)$ is some preassigned weight function.

If a measure W_n^2 is adopted, the hypothesis is rejected for those samples for which $W_n^2 > z_1$, say, and if a measure K_n is adopted, the hypothesis is rejected when $K_n > z_2$, say. The numbers z_1 and z_2 are to be chosen so that when the hypothesis is true the probability of rejection is some specified number (for

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example, .01 or .05). The main purpose of this paper is to give methods for finding the asymptotic distributions of W_n^2 and K_n, and, hence, approximate values of the significance points, z_1 and z_2. We assume that the hypothetical distribution is continuous.

The fundamental ideas for tests of this nature are due to Kolmogorov [11], Smirnov [17], Cramér [2], and von Mises [19], and for large n certain tests have been developed by them. The present paper treats these tests in somewhat more detail, the analysis being greatly expedited by reducing the problems to straightforward considerations in the theory of continuous Gaussian stochastic processes. This reduction was developed by Doob [6], and used by him to give a simplified proof of Kolmogorov's fundamental result.

The principal innovation in this paper is the incorporation of a weight function to allow more flexibility in the tests. Although we are able to make explicit calculations for only a few simple types of weight functions, the principal mathematical problems are reduced to classical problems in the theory of differential equations.

The function $\psi(t)$, $0 \leq t \leq 1$, is to be chosen by the statistician so as to weight the deviations according to the importance attached to various portions of the distribution function. This choice depends on the power against the alternative distributions considered most important. The selection of $\psi(t) \equiv 1$ yields $n\omega^2$, the criterion of von Mises, for W_n^2, and the criterion of Kolmogorov for K_n. For W_n^2 to exist for all samples except a set with probability zero, it is necessary and sufficient that the following integrals exist:

$$(2.3) \qquad \int_0^{u_1} u^2\psi(u)\,du$$

for every $u_1 (0 < u_1 < 1)$,

$$(2.4) \qquad \int_{u_2}^{1} (1 - u)^2\psi(u)\,du$$

for every u_2 $(0 < u_2 < 1)$.

Given the data $x_1, x_2, \cdots, x_n$ arranged in increasing magnitude (with probability one there are no equalities between any two of them, since the distribution is assumed continuous), we obtain for practical computations the simpler variants of (2.1) and (2.2),

$$(2.5) \quad W_n^2 = 2\sum_{j=1}^{n}\left\{\phi_2\,[F(x_j)] - \frac{2j-1}{2n}\,\phi_1[F(x_j)]\right\} + n\int_0^1 (1 - t)^2\psi(t)\,dt,$$

$$(2.6) \quad K_n = \frac{1}{\sqrt{n}} \max_{j=1,\cdots,n} \{\sqrt{\psi[F(x_j)]}\,\max\,[nF(x_j) - (j - 1), j - nF(x_j)]\},$$

where

$$(2.7) \qquad \phi_1(t) = \int_0^t \psi(s)\,ds, \qquad \phi_2(t) = \int_0^t s\psi(s)\,ds.$$

For (2.5) to hold the integrals $\phi_1(t)$, $\phi_2(t)$ must exist; for (2.6) to hold it is necessary and sufficient that

$$\frac{1}{\psi(t)}\left|\frac{d}{dt}[t(1-t)\psi(t)]\right| \leqq 1 \tag{2.8}$$

if $\psi(t)$ is differentiable (substituting the difference quotient in (2.8) if $\psi(t)$ is not differentiable).

3. Reduction to a continuous stochastic process. Since $F(x)$ is assumed continuous, we can make the transformation $u = F(x)$. Then the observations are $u_i = F(x_i)$ $(i = 1, 2, \cdots, n)$, and under the null hypothesis these can be considered as drawn from the uniform distribution between 0 and 1. Let $G_n(u)$ be the empirical distribution derived from $u_1, \cdots, u_n$. Then W_n^2 and K_n are, respectively,

$$W_n^2 = n\int_0^1 [G(u) - u]^2\psi(u)\,du, \tag{3.1}$$

$$K_n = \sup_{0\leqq u\leqq 1} \sqrt{n}\,|G_n(u) - u|\sqrt{\psi(u)}. \tag{3.2}$$

For every $0 \leqq u \leqq 1$, $Y_n(u) = \sqrt{n}[G_n(u) - u)]$ is a random variable and the set of these random variables may be considered a stochastic process with parameter u. Putting

$$A_n(z) = \Pr\left\{\int_0^1 Y_n^2(u)\psi(u)du \leqq z\right\} = \Pr\{W_n^2 \leqq z\}, \tag{3.3}$$

$$B_n(z) = \Pr\{\sup_{0\leqq u\leqq 1} |Y_n(u)|\sqrt{\psi(u)} \leqq z\} = \Pr\{K_n \leqq z\}, \tag{3.4}$$

we wish to calculate $A(z) = \lim A_n(z)$, $n \to \infty$, and $B(z) = \lim B_n(z)$, $n \to \infty$, if these limits exist.

For fixed $u_1, u_2, \cdots, u_k$ the joint distribution of $Y_n(u_1), Y_n(u_2), \cdots, Y_n(u_k)$ approaches a k-variate normal distribution as $n \to \infty$. Thus the asymptotic process is Gaussian (normal) and is specified by its mean and covariance functions. For finite n we have

$$\begin{aligned} E(Y_n(u)) &= 0, \\ E(Y_n(u)Y_n(v)) &= \min(u, v) - uv. \end{aligned} \tag{3.5}$$

The limiting process is a Gaussian process $y(u)$, $0 \leqq u \leqq 1$, for which

$$\begin{aligned} E(y(u)) &= 0, \\ E(y(u)y(v)) &= \min(u, v) - uv, \end{aligned} \tag{3.6}$$

such that the probability is 1 that $y(u)$ is continuous [6]. Putting

$$a(z) = \Pr\left\{\int_0^1 y^2(u)\psi(u)\,du \leqq z\right\}, \tag{3.7}$$

$$b(z) = \Pr\{\sup_{0 \leqq u \leqq 1} |y(u)| \sqrt{\psi(u)} \leqq z\}, \tag{3.8}$$

we wish to conclude that $A(z) = a(z)$ and $B(z) = b(z)$. Having established these equalities we shall be in a position to use certain representation theorems for stochastic processes to great advantage.

In [4] Donsker has given the following theorem: *Let R be the space of real, single-valued functions g(t) which are continuous except for at most a finite number of finite jumps, and let C be the space of continuous functions. Let F(g) be a functional defined on R and continuous in the uniform topology, i.e.,* $\sup_{0 \leqq t \leqq 1} |g_n(t) - g_0(t)| \to 0$, $n \to \infty$, *implies* $|F(g_n) - F(g_0)| \to 0$, $n \to \infty$, $g_n \in R$, $g_0 \in C$, *except for a set of* $g_0(t)$ *with* 0 *probability according to the probability associated with* $y(t)$. *Then*

$$\lim_{n\to\infty} \Pr\{F[Y_n(t)] \leqq z\} = \Pr\{F[y(t)] \leqq z\}. \tag{3.9}$$

It follows from this result that if $\psi(u)$ is bounded $A(z) = a(z)$ and $B(z) = b(z)$.

To handle more general weight functions for the case of integrals we want to extend this result. We shall assume that $\psi(u)$ is continuous in any interval $0 < u_1 \leqq u \leqq u_2 < 1$. Secondly we assume that

$$\int_0^{u_1} \psi(t) t \log\log \frac{1}{t}\, dt, \qquad \int_{u_1}^1 \psi(t)(1 - t) \log\log \frac{1}{1-t}\, dt \tag{3.10}$$

exist for every u_1 $(0 < u_1 < 1)$. It is shown in Section 5 that

$$(1 + t)y(t/(1 + t)) = X(t)$$

is the Wiener process which has the property ([12] p. 242 and p. 247)

$$\Pr\left\{\text{there exists a } t_0 \text{ such that } X^2(t) \leqq 2t \log\log \frac{1}{t} \text{ for } 0 < t < t_0\right\} = 1. \tag{3.11}$$

This implies that

$$\Pr\left\{\begin{array}{l}\text{there exists a } u_0 \text{ such that} \\ y^2(u) \leq 2u(1-u) \log\log \dfrac{1-u}{u} \text{ for } 0 < u < u_0\end{array}\right\} = 1. \tag{3.12}$$

Thus with probability 1 $\psi(t)y^2(t)$ is majorized by $k\psi(t)t\log\log(1/t)$ for $k \geqq 2(1 - u_0)$. Thus if the first integral in (3.10) exists

$$\int_0^{u_1} y^2(t)\psi(t)\, dt \tag{3.13}$$

exists with probability 1 (taking the principal value when the integral is improper). A similar argument holds for the existence of

$$\int_{u_1}^1 y^2(t)\psi(t)\, dt. \tag{3.14}$$

Thus $\int_0^1 y^2(t)\psi(t)\,dt$ exists with probability 1. This defines a functional continuous in the uniform topology. Hence from Donsker's theorem $A(z) = a(z)$.

4. The limiting distribution of the integral criterion. In this section we show how to find $a(z)$ in terms of the solution of a certain differential equation and give two examples of this method. The statistic W_n^2 is essentially that introduced by Cramér [2]; in the case of $\psi(t) \equiv 1$, it is n times the ω^2 criterion studied by von Mises [19] and Smirnov [17].

The method we use is analogous to the technique of Kac and Siegert [10]. We shall sketch briefly the extension of their results.

By Mercer's theorem a symmetric continuous correlation function $k(s, t)$, $0 \leqq s, t \leqq 1$, which is square integrable (in one variable and in both variables), can be expressed as

$$k(s, t) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} f_j(s) f_j(t), \tag{4.1}$$

where λ_i is an eigenvalue and $f_i(t)$ is the corresponding normalized eigenfunction of the integral equation

$$\lambda \int_0^1 k(s, t) f(s)\,ds = f(t), \tag{4.2}$$

and

$$\int_0^1 f_i(t) f_j(t)\,dt = \delta_{ij}, \tag{4.3}$$

the Kronecker delta. In most cases $k(0, 0) = k(1, 1) = 0$; hence $f_i(0) = f_i(1) = 0$. Since $k(s, t)$ is positive definite, $\lambda_i > 0$. The series (4.1) converges absolutely and uniformly in the unit square.

Let $X_1, X_2, \cdots$ be independently, normally distributed with means zero and variances 1. If $k(t, t) < \infty$, then we can define

$$z(t) = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} X_j f_j(t); \tag{4.4}$$

the series converges in the mean and with probability one for each t. Then $z(t)$ is a Gaussian process with $Ez(t) = 0$ and $Ez(s)z(t) = k(s, t)$. Thus $z(t)$ gives the same stochastic process as $\sqrt{\psi(t)}\, y(t)$ when $k(s, t) = \sqrt{\psi(s)}\sqrt{\psi(t)}$ [min $(s, t) - st$]. From this it follows that with probability 1

$$\begin{aligned} W^2 &= \int_0^1 \psi(t) y^2(t)\,dt = \int_0^1 z^2(t)\,dt = \int_0^1 \left[\sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} f_j(t) X_j\right]^2 dt \\ &= \sum_{j=1}^{\infty} \frac{1}{\lambda_j} X_j^2. \end{aligned} \tag{4.5}$$

See [10] for details of this proof. Thus

$$
\begin{aligned}
E[e^{iuW^2}] &= E\left[\exp\left(iu\sum_{j=1}^{\infty} X_j^2/\lambda_j\right)\right] \\
&= \prod_{j=1}^{\infty} E[\exp iuX_j^2/\lambda_j] \qquad (4.6)\\
&= \prod_{j=1}^{\infty} (1 - 2iu/\lambda_j)^{-\frac{1}{2}}.
\end{aligned}
$$

The infinite product converges absolutely and uniformly for all real u, and in general $1/\lambda_n = O(1/n^2)$.

We desire a more general result, however, because one weight function we treat leads to a kernel that is not continuous at (0, 0) and (1, 1). We use the following theorem of Hammerstein [9]: *Let $k(s, t)$ be continuous in the unit square except possibly at the corners of the square; let $\partial k(s, t)/\partial s$ be continuous in the interior of both triangles in which the square is divided by the line between* (0, 0) *and* (1, 1), *and let the partial derivative be bounded in the domain $\epsilon \leqq s \leqq 1 - \epsilon$ and $0 \leqq t \leqq 1$ for each $\epsilon(> 0)$. Then the series on the right of* (4.1) *converges uniformly to $k(s, t)$ in every domain in the interior of the unit square.*

Since $k(s, t) = \sqrt{\psi(s)}\sqrt{\psi(t)}\,[\min(s, t) - st]$, the condition is that $\psi(t)$ be continuous for $0 < t < 1$ and

$$
\sqrt{\frac{\psi(t)}{\psi(s)}}\; t[\tfrac{1}{2}(1 - s)\psi'(s) - \psi(s)] \tag{4.7}
$$

be continuous for $0 \leqq t \leqq s \leqq 1 - \epsilon$ and

$$
\sqrt{\frac{\psi(t)}{\psi(s)}}\; (1 - t)[\tfrac{1}{2}s\psi'(s) + \psi(s)] \tag{4.8}
$$

be continuous for $\epsilon \leqq s \leqq t \leqq 1$.

In this case (4.4) converges in the mean and with probability one for every $t(\epsilon \leqq t \leqq 1 - \epsilon)$, and $z(t)$ is the same process as $x(t)$ in this interval. If $\int_0^1 k(t, t)\,dt < \infty$, $\sum_{j=1}^{\infty} 1/\lambda_j < \infty$ (by Bessel's inequality) and $\sum_{j=1}^{\infty} X_j^2/\lambda_j$ converges with probability one. Further, with probability one, $\sum_{j=1}^{\infty} X_j f_j(t)/\sqrt{\lambda_j}$ converges in the mean (integral with respect to t) and it converges to $z(t)$. Thus we have with probability one

$$
\int_0^1 z^2(t)\,dt = \sum_{j=1}^{\infty} X_j^2/\lambda_j, \tag{4.9}
$$

$$
\int_\epsilon^{1-\epsilon} x^2(t)\,dt = \int_\epsilon^{1-\epsilon} z^2(t)\,dt = \int_\epsilon^{1-\epsilon}\left[\sum_{j=1}^{\infty}\frac{1}{\sqrt{\lambda_j}} X_j f_j(t)\right]^2 dt. \tag{4.10}
$$

For ϵ small enough

$$
E\left[\int_0^1 x^2(t)\,dt + \int_\epsilon^{1-\epsilon} x^2(t)\,dt\right] = \int_0^\epsilon k(t, t)\,dt + \int_{1-\epsilon}^1 k(t, t)\,dt < \delta \tag{4.11}
$$

for any $\delta > 0$. Thus the distribution of $W^2 = \int_0^1 x^2(t)\, dt$ is the limiting distribution of $\int_\epsilon^{1-\epsilon} x^2(t)\, dt$. With a similar argument for the integral of $z^2(t)$ we argue that the distribution of W^2 is the distribution of $\sum_{j=1}^\infty X_j^2/\lambda_j$ with characteristic function (4.6).

THEOREM 4.1. *If*

$$k(s, t) = \sqrt{\psi(s)}\sqrt{\psi(t)}\,[\min(s, t) - st] \tag{4.12}$$

is continuous or if $k(s, t)$ *is continuous except at* $(0, 0)$ *and* $(1, 1)$ *with* $\partial k(s, t)/\partial s$ *continuous for* $0 < s, t < 1$, $s \neq t$, *and bounded in* $\epsilon \leqq s \leqq 1 - \epsilon$, $0 \leqq t \leqq 1$ *for every* ϵ (>0) *then the characteristic function of* W^2 *is given by* (4.6), *where* $\{\lambda_j\}$ *are the eigenvalues of* $k(s, t)$ *defined by* (4.2).

In our case the integral equation is

$$f(t) = \lambda \int_0^1 [\min(t, s) - ts]\sqrt{\psi(t)}\sqrt{\psi(s)}\, f(s)\, ds. \tag{4.13}$$

It can be shown that if $f(t)$ satisfies (4.13) for some λ, then $h(t) = f(t)\psi^{-\frac{1}{2}}(t)$ satisfies

$$h''(t) + \lambda\psi(t)h(t) = 0 \tag{4.14}$$

for that λ (see [8], Sections 604 and 605) and $h(0) = h(1) = 0$ when $k(0, 0) = k(1, 1) = 0$. Let $h(t, \lambda)$ be the solution of (4.14) for which

$$h(0, \lambda) = 0, \qquad \left.\frac{\partial h(t, \lambda)}{\partial t}\right|_{t=0} = 1. \tag{4.15}$$

If $\psi(t)$ is continuous $(0 \leqq t \leqq 1)$, such a solution exists and $h(t, \lambda)$ is continuous in $t(0 \leqq t \leqq 1)$. Since $h(1, \lambda) = 0$ for λ an eigenvalue of (4.13), the roots of $h(1, \lambda) = 0$ are the roots of the Fredholm determinant $D(\lambda)$ associated with $k(s, t)$. It can be shown that

$$D(\lambda) = \frac{h(1, \lambda)}{h(1, 0)} = \prod_{i=1}^\infty \left(1 - \frac{\lambda}{\lambda_i}\right). \tag{4.16}$$

The characteristic function (4.6) is

$$\frac{1}{\sqrt{D(2it)}}. \tag{4.17}$$

The square root is taken so as to make (4.17) real and positive when the characteristic function is real and positive. The details of this proof are given in [8], Section 605.

THEOREM 4.2. *Let* $\psi(t)$ *be continuous for* $0 \leqq t \leqq 1$. *Then the equation* (4.14) *has a unique solution* $h(t, \lambda)$ *for every* $\lambda > 0$ *satisfying* (4.15). *Then the characteristic function of* W^2 *is*

$$\sqrt{\frac{h(1, 0)}{h(1, 2it)}}. \tag{4.18}$$

Thus we have reduced the problem of finding the characteristic function of W^2 to finding the general solution of a differential equation.

The semi-invariants κ_n of W^2 are given quite easily (when they exist) through the eigenvalues. Since

$$\phi(t) = \prod_{j=1}^{\infty} (1 - 2it/\lambda_j)^{-\frac{1}{2}}, \tag{4.19}$$

the coefficient of $(it)^n/n!$ in the power series expansion of $\log \phi(t)$ is

$$\kappa_n = 2^{n-1}(n-1)! \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_j}\right)^n, \qquad n = 1, 2, \cdots. \tag{4.20}$$

Hence we obtain for the mean and variance, for instance,

$$\begin{aligned} \kappa_1 &= \mu = \sum \frac{1}{\lambda_j}, \\ \kappa_2 &= \sigma^2 = 2 \sum \left(\frac{1}{\lambda_j}\right)^2. \end{aligned} \tag{4.21}$$

Even without knowing the eigenvalues, the moments can be calculated in terms of the iterates of the kernel $k(s, t)$. Putting $k_1(s, t) = k(s, t) = (\min(s, t) - st)\sqrt{\psi(s)\psi(t)}$, $k_{n+1}(s, t) = \int_0^1 k_n(s, u)k(u, t)\, du$, we have by means of the bilinear expansion

$$k_n(s, t) = \sum \lambda_j^{-n} f_j(s) f_j(t). \tag{4.22}$$

Hence,

$$\kappa_n = 2^{n-1}(n-1)! \int_0^1 k_n(s, s)\, ds \tag{4.23}$$

and, in particular,

$$\begin{aligned} \mu &= \int_0^1 k(s, s)\, ds = \int_0^1 s(1-s)\psi(s)\, ds, \\ \sigma^2 &= 2\int_0^1 \int_0^1 k^2(s, t)\, ds\, dt = 4\int_0^1 (1-s)^2 \psi(s) \int_0^s t^2 \psi(t)\, dt\, ds. \end{aligned} \tag{4.24}$$

We now present two applications of this method.

Example 1. Let $\psi(t) \equiv 1$; then $W_n^2 = n\omega^2$. The differental equation $h''(t) + \lambda h(t) = 0$ has a solution

$$h(t, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda} t \tag{4.25}$$

satisfying (4.15). Taking $h(1, 0)$ as $\lim_{\lambda \to 0} h(1, \lambda) = 1$, we find that $1/\sqrt{D(2it)}$ is

$$\phi_1(t) = Ee^{itW^2} = \sqrt{\frac{\sqrt{2it}}{\sin \sqrt{2it}}} = \sqrt{\frac{\sqrt{-2it}}{\sinh \sqrt{-2it}}}. \tag{4.26}$$

This expression was given by Smirnov [15] and later by von Mises [20] using entirely different methods. A formal method for finding the distribution (by inverting the Fourier transform) was given later by Smirnov [16], but his expression is not amenable to numerical calculation. The following procedure expresses $a_1(z) = \Pr\{W^2 \leqq z\}$ in terms of tabulated functions.

It appears convenient to work with the Laplace transform. We have

$$\xi(t) = \phi_1(it) = E(e^{-tW^2}) = \sqrt{\frac{\sqrt{2t}}{\sinh \sqrt{2t}}}. \tag{4.27}$$

Using integration by parts, we obtain

$$\int_0^\infty e^{-zt} a_1(z)\, dz = \frac{1}{t}\xi(t) \tag{4.28}$$

for the cdf $a_1(z)$. We wish to invert this Laplace transform. Now

$$\frac{1}{t}\xi(t) = \left(\frac{2}{t}\right)^{3/4} e^{-\frac{1}{2}\sqrt{2t}}(1 - e^{-2\sqrt{2t}})^{-\frac{1}{2}}. \tag{4.29}$$

We suppose in the sequel that the real part of t, $R(t) > 0$ and apply the binomial expansion to the last expression; thus

$$\frac{\xi(t)}{t} = \left(\frac{2}{t}\right)^{3/4} \sum_{j=0}^\infty (-1)^j \binom{-\frac{1}{2}}{j} e^{-(2j+\frac{1}{2})\sqrt{2t}}, \tag{4.30}$$

where $\binom{-\frac{1}{2}}{j} = (-1)^j\Gamma(j + \frac{1}{2})/[\Gamma(\frac{1}{2})j!]$. It may be readily verified that the complex inversion formula can be used termwise here since the abscissa of convergence of $\xi(t)/t$ is $R(t) = 0$, and the above series converges absolutely and uniformly in the half plane $R(t) \geqq \beta > 0$.

Since

$$\begin{aligned} e^{-A\sqrt{t}} &= \int_0^\infty e^{-st} \frac{A}{2\sqrt{\pi}\, s^{3/2}} e^{-A^2/(4s)}\, ds, \\ \frac{1}{t^{3/4}} &= \int_0^\infty e^{-st} \frac{ds}{\Gamma(3/4)s^{1/4}}, \end{aligned} \tag{4.31}$$

we have

$$\frac{e^{-A\sqrt{t}}}{t^{3/4}} = \int_0^\infty e^{-zt}\phi(z)\, dz, \tag{4.32}$$

where

$$\phi(z) = \frac{A}{2\sqrt{\pi}\,\Gamma(3/4)} \int_0^z \frac{e^{-A^2/(4x)}}{x^{3/2}(z - x)^{1/4}}\, dx \tag{4.33}$$

by virtue of the convolution property of the Laplace transform. In this integral we change variables, putting $x = u \operatorname{sech}^2 \theta$ to give

$$\phi(z) = \frac{A}{\sqrt{\pi}\,\Gamma(3/4) z^{3/4}} \int_0^\infty e^{-(A^2/(4z)) \cosh^2 \theta} (\cosh \theta \sinh \theta)^{\frac{1}{2}} \, d\theta$$

$$(4.34) \qquad = \frac{A e^{-A^2/(8z)}}{2^{3/2} \sqrt{\pi}\,\Gamma(3/4) z^{3/4}} \int_0^\infty e^{-(A^2/(8z)) \cosh \theta} (\sinh \theta)^{\frac{1}{2}} \, d\theta$$

$$= \frac{e^{-A^2/(8z)}}{\sqrt{2}\,\pi} \sqrt{\frac{A}{z}}\, K_{\frac{1}{4}}\left(\frac{A^2}{8z}\right),$$

where $K_{\frac{1}{4}}(x)$ is the standard Bessel function [22].

Having inverted the typical term, we finally obtain by summing

$$(4.35) \qquad a_1(z) = \frac{1}{\pi\sqrt{z}} \sum_{j=0}^{\infty} (-1)^j \binom{-\frac{1}{2}}{j} \cdot (4j+1)^{\frac{1}{2}} e^{-(4j+1)^2/(16z)} K_{\frac{1}{4}}((4j+1)^2/(16z)).$$

The convergence of this series is very rapid. If $a_1(z) = \sum_{j=0}^{\infty} u_j(z)$, we find that $u_{j+1}(z)/u_j(z) < k_j e^{-(4j+1)/(2z)}$ (using the fact that $K_{\frac{1}{4}}(t)$ is a decreasing function of t), where $k_0 < 1.12$, $k_1 < 1.007$, $k_2 < 1.002$, $k_j < 1.0007$ for $j \geqq 3$. Since $K_{\frac{1}{4}}(t)$ is positive, $u_j(z) > 0$. Using a crude geometric series bound for $R_4(z) = \sum_{j=4}^{\infty} u_j(z)$, we can show that for $z \leqq 2$, $R_4(z) < .0002$. Moreover, for $z \leqq 2$, $R_4(z) < u_3(z) < u_2(z) < u_1(z)$. In computation, therefore, one need only take as many terms in the series as are different from 0 in the number of decimal places carried. We give below a table of z for equal increments (.01) of $a_1(z)$ with the 5%, 1% and .1% significance points. The calculations have been carried to 6 figures before rounding off. The authors are indebted to Mr. Jack Laderman of Columbia University and the Numerical Analysis Department of the Rand Corporation for their assistance in preparing the table.

The semi-invariants of this distribution are easily obtained since the eigenvalues are $\lambda_j = 1/(\pi^2 j^2)$. Thus

$$(4.36) \qquad \kappa_n = \frac{2^{n-1}(n-1)!}{\pi^{2n}} \sum_{j=1}^{\infty} \frac{1}{j^{2n}} = 2^{3n-2} \frac{(n-1)!}{(2n)!} B_n,$$

where B_n are the Bernoulli numbers: $B_1 = 1/6$, $B_2 = 1/30$, etc.

Example 2. $\psi(t) = 1/[t(1-t)]$. Since the variance of $Y_n(t) = \sqrt{n}\,[G_n(t) - t]$ is $t(1-t)$, an interesting weight function for $Y_n^2(t)$ is the reciprocal of this variance.[2] In a certain sense, this function assigns to each point of the distribution

[2] This suggestion was first made by L. J. Savage.

TABLE 1

Limiting Distribution of $n\omega^2$

$a_1(z) = \lim_{n \to \infty} \Pr\{n\omega^2 \leq z\}$

z	$a_1(z)$	z	$a_1(z)$	z	$a_1(z)$
.02480	.01	.08562	.34	.17159	.67
.02878	.02	.08744	.35	.17568	.68
.03177	.03	.08928	.36	.17992	.69
.03430	.04	.09115	.37	.18433	.70
.03656	.05	.09306	.38	.18892	.71
.03865	.06	.09499	.39	.19371	.72
.04061	.07	.09696	.40	.19870	.73
.04247	.08	.09896	.41	.20392	.74
.04427	.09	.10100	.42	.20939	.75
.04601	.10	.10308	.43	.21512	.76
.04772	.11	.10520	.44	.22114	.77
.04939	.12	.10736	.45	.22748	.78
.05103	.13	.10956	.46	.23417	.79
.05265	.14	.11182	.47	.24124	.80
.05426	.15	.11412	.48	.24874	.81
.05586	.16	.11647	.49	.25670	.82
.05746	.17	.11888	.50	.26520	.83
.05904	.18	.12134	.51	.27429	.84
.06063	.19	.12387	.52	.28406	.85
.06222	.20	.12646	.53	.29460	.86
.06381	.21	.12911	.54	.30603	.87
.06541	.22	.13183	.55	.31849	.88
.06702	.23	.13463	.56	.33217	.89
.06863	.24	.13751	.57	.34730	.90
.07025	.25	.14046	.58	.36421	.91
.07189	.26	.14350	.59	.38331	.92
.07354	.27	.14663	.60	.40520	.93
.07521	.28	.14986	.61	.43077	.94
.07690	.29	.15319	.62	.46136	.95
.07860	.30	.15663	.63	.49929	.96
.08032	.31	.16018	.64	.54885	.97
.08206	.32	.16385	.65	.61981	.98
.08383	.33	.16765	.66	.74346	.99
				1.16786	.999

$F(x)$ equal weights. A statistician may prefer to use this weight function when he feels that $\psi(t) \equiv 1$ does not give enough weight to the tails of the distribution.

In this example

$$(4.37)\qquad \begin{aligned} k(t, s) &= \sqrt{\frac{t(1-s)}{(1-t)s}}, \qquad t \leqq s, \\ &= \sqrt{\frac{(1-t)s}{t(1-s)}}, \qquad t \geqq s, \end{aligned}$$

is not continuous at $(t, s) = (0, 0)$ or $(1, 1)$; hence we need the extended result of Theorem 4.1 to justify our procedure. It is known that the Ferrer associated Legendre polynomials $f_i(t) = P_i^1(t) = t(1 - t)P_i(2t - 1)$ satisfy the integral equation with $\lambda_i = 1/[i(i + 1)]$ (see [23], p. 324). Thus the characteristic function of W^2 is

$$(4.38)\qquad \begin{aligned} \phi_2(t) &= \prod_{j=1}^{\infty}\left(1 - \frac{2it}{j(j+1)}\right)^{-\frac{1}{2}} \\ &= \sqrt{\frac{-2\pi it}{\cos\left(\frac{\pi}{2}\sqrt{1+8it}\right)}}. \end{aligned}$$

An analysis similar to that used in Example 1 shows that the cdf, $a_2(z)$, can be expressed as

$$\begin{aligned} a_2(z) &= \Pr\{W^2 \leqq z\} \\ &= \sqrt{\frac{\pi}{2}}\,\frac{1}{z}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}(4j+1)\int_0^1 e^{(rz)/8-((4j+1)^2\pi^2)/(8rz)}\,\frac{dr}{r^{3/2}(1-r)^{1/2}} \\ &= \frac{\sqrt{2\pi}}{z}\sum_{j=0}^{\infty}\binom{-\frac{1}{2}}{j}(4j+1)e^{-((4j+1)^2\pi^2)/(8z)}\int_0^{\infty} e^{z/(8(w^2+1))-((4j+1)^2\pi^2w^2)/(8z)}\,dw. \end{aligned}$$

5. Theory of deviations. The second test criterion led to the calculation of

$$B_n(z) = \Pr\{\sup_{0 \leqq u \leqq 1} \sqrt{n}\,|\,G_n(u) - u\,|\,\sqrt{\psi(u)} \leqq z\}.$$

In order to handle the limiting distribution we consider the functional

$$(5.1)\qquad K = \sup_{0 \leqq u \leqq 1} |\,y(u)\,|\,\sqrt{\psi(u)}.$$

It follows from the theorem of Donsker [4] that for $\psi(u)$ bounded we have

$$\lim_{n\to\infty} B_n(z) = \Pr\{K \leqq z\},$$

and the problem is reduced to that of calculating the distribution of (5.1). This is the elegant idea of Doob [6], who treated the case $\psi \equiv 1$.

This is known as an "absorption probability" problem on account of its very suggestive analogy with a simple diffusion model. It is clear that the event that $\{-z(\psi(u))^{-\frac{1}{2}} \leqq y(u) \leqq z(\psi(u))^{-\frac{1}{2}}, 0 \leqq u \leqq 1\}$ is equivalent to the event $\{K \leqq z\}$; thus the probability $b(z)$ is, very crudely speaking, the "proportion" of all those paths $y(u)$ of the diffusing particle which do not get "absorbed into" (i.e., intersect) the "barriers" $y = \pm z(\psi(u))^{-\frac{1}{2}}$.

It is convenient to make a transformation due to Doob [6] which renders the analysis simpler. If we put

$$X(t) = (1 + t)y\left(\frac{t}{1 + t}\right),$$

it is easy to verify that $X(t)$ is the Wiener-Einstein process; that is, $X(t)$ is Gaussian, $X(0) = 0$, $E(X(t)) = 0$, $E(X(t)X(s)) = \min(s, t)$. Then

$$b(z) = \Pr\{|X(t)| \leqq \xi(t), 0 \leqq t \leqq \infty\},$$

where

$$\xi(t) = \frac{z(1 + t)}{\sqrt{\psi\left(\frac{t}{1 + t}\right)}}. \tag{5.2}$$

Thus we have the absorption probability problem for the free particle with barriers $x = \pm\xi(t)$ for $t \geqq 0$.

The method of solution is to treat the corresponding diffusion problem as a boundary value problem with the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \tag{5.3}$$

associated with the region $t \geqq 0$, $|x| \leqq \xi(t)$. In line with the preceding analogy $f(t, x)$ will be the "density" of paths $X(u)$ which for $0 \leqq u \leqq t$ have not been "absorbed" and for which $X(t) = x$; hence

$$\int_{|x|<\xi(t)} f(t, x)\, dx$$

will give the probability of nonabsorption up to time t. It is the limit of this expression for $t \to \infty$ which will yield $b(z)$. For a more detailed discussion of these points see Lévy [12], pp. 78 et seq.

We need the following existence and uniqueness theorem:

THEOREM 5.1. *Given that $\xi(t)$ of (5.2) has a bounded derivative for $t_0 \leqq t \leqq t_1$, there exists a unique function $p(t_0, y; t, x)$ such that for any continuous function $g(y)$, $|y| < \xi(t_0)$, the function*

$$f(t, x) = \int_{|y|<\xi(t_0)} g(y)p(t_0, y; t, x)\, dy \tag{5.4}$$

has the following properties:

(1) *$f(t, x)$ satisfies (5.3) in the domain $t_0 < t < t_1$, $|x| < \xi(t)$,*

(2) $$\lim_{x\to\pm\xi(t)} f(t, x) = 0, \qquad t_1 > t > t_0,$$

(3) $$\lim_{\substack{t \to t_0 \\ x \to \eta}} f(t, x) = g(\eta) \qquad |\eta| < \xi(t_0).$$

The proof of this theorem is contained quite explicitly in the fundamental paper of Fortet [7] (especially ch. V), who considers in great detail the general problem of absorption probabilities. Fortet treats only the case of one absorbing barrier, but his results are easily extended to the above case of two barriers. The differential $p(t_0, y; t, x)\, dx$ can be interpreted, to terms of order $(dx)^2$, as the probability that if the diffusing particle starts at (t_0, y) it will not have been absorbed in the barriers $\pm\xi(t)$ during the interval (t_0, t), and will lie between x and $x + dx$ at time t.

We have not stated the best theorem possible. If $\xi(t)$ is merely continuous the absorption probability density $f(t, x)$ exists. For the existence of a solution to (5.3) satisfying (2) and (3) of Theorem 5.1 it is sufficient to require that $\xi(t)$ satisfy a Lipschitz condition associated with the law of the iterated logarithm. Finally we remark in passing that unless $f(t, x)$ is of the form (5.4) (the so called "normal" solution of Fortet) its uniqueness is not assured (cf. Doetsch [3]).

If in the theorem $\xi(t)$ has a bounded derivative for $t \geqq 0$ then we plainly have

$$b(z) = \lim_{t \to \infty} \int_{-\xi(t)}^{\xi(t)} p(0, 0; t, x)\, dx, \tag{5.5}$$

but if $\xi(t)$ does not have a bounded derivative for $t \geqq 0$, (5.5) can no longer be employed to determine $b(z)$. However, if there are a finite number of intervals in each of which $\xi(t)$ has a bounded derivative and between which $\xi(t)$ has a simple jump discontinuity it is easy to modify the above result; in fact over some of the intervals $\xi(t)$ may be infinite. A piecewise determination can be made and the solution can be continued to beyond the last discontinuity, and then (5.5) can be used. Suppose the points of discontinuity of $\xi(t)$ are $0 < t_1 < t_2 < \cdots < t_n$ and suppose $\xi(t)$ is, say, left continuous. In the region $(0, t_1)$ we have the solution $g_0(t, x) = p_0(0, 0; t, x)$ by the above theorem. Now if $\xi(t_1) < \xi(t_1 + 0)$ we define $g_1^*(t_1, x)$ by

$$g_1^*(t_1, x) = \begin{cases} g_0(t_1, x), & |x| \leqq \xi(t_1), \\ 0, & \xi(t_1) \leqq |x| \leqq \xi(t_1 + 0), \end{cases}$$

and if $\xi(t_1) > \xi(t_1 + 0)$ we define $g_1^*(t_1, x) = g_0(t_1, x)$, $|x| \leqq \xi(t_1 + 0)$. Then $g_1^*(t_1, x)$ is continuous in $|x| < \xi(t_1 + 0)$ and we have for $t_1 < t < t_2$ a function $g_1(t, x)$ defined by Theorem 5.1;

$$g_1(t, x) = \int_{|y| < \xi(t_1+0)} g_1^*(t_1, y) p_1(t_1, y; t, x)\, dy.$$

In the same way we can define a function $g_2^*(t_2, x)$ which will yield a function

$g_2(t, x)$ for $t_2 < t < t_3$. This process will ultimately yield a unique function $g_n(t, x)$ for $t > t_n$. Finally

$$b(z) = \lim_{t\to\infty} \int_{-\xi(t)}^{\xi(t)} g_n(t, x)\, dx. \tag{5.6}$$

It is clear that if $\xi(t) = \infty$ in some of the intervals the successive determination of the functions $g_k(t, x)$ may still be carried forward. This would correspond to an absence of the absorbing barrier over the interval.

Using the relation (5.2) and the above remarks we have the following theorem for the weight function $\psi(u)$:

THEOREM 5.2. *Suppose there is a finite sequence* $0 = u_0 < u_1 < u_2 \cdots < u_n < u_{n+1} = 1$ *such that in the interval* $(u_k, u_{k+1}]$ $\psi(t)$ *is either* (1) *identically zero or* (2) *is bounded away from zero and has a bounded derivative. Then there exists a unique sequence of functions* $\{p_k(t_k, y; t, x)\}$ *such that for* t *in the interval* $((u_k/(1 - u_k) = t_k < t < t_{k+1} = u_{k+1}/(1 - u_{k+1}))$ *the conclusions of Theorem* 5.1 *hold for the functions* $p_k(t_k, y; t, x)$, $k = 0, 1, \cdots, n$, $\xi(t)$ *being defined by* (5.2).

From this theorem we can generate a set of functions $g_k(t, x)$, $t_k < t < t_{k+1}$, $k = 0, 1, \cdots, n$, and another set $g_k^*(t_k, x)$, $k = 1, 2, \cdots, n$, as before. $g_{k+1}^*(t_{k+1}, x)$ agrees with $g_k(t_{k+1}, x)$ over the set of x for which the latter is defined; that is, $|x| < \xi(t_{k+1})$, and is zero for other values; namely, $\xi(t_{k+1} + 0) > |x| > \xi(t_{k+1})$ if $\xi(t)$ has a positive jump at t_{k+1}. Putting

$$g_0(t, x) = p_0(0, 0; t, x), \qquad t \leqq t_1,$$

$$g_k(t, x) = \int_{|y| < \xi(t_k+0)} g_k^*(t_k, y) p_k(t_k, y; t, x)\, dy, \quad t_k < t < t_{k+1}, \quad k = 1, 2, \ldots n,$$

we finally have (5.6) for $b(z)$.

In a formal way the problem is thus solved, but the analytical difficulties of getting an explicit solution may be prohibitive. If $\xi(t)$ consists of a set of linear arcs (which implies that $\sqrt{\psi(u)}$ is of the form $(\alpha u + \beta)^{-1}$ in a piecewise way) then $b(z)$ can be determined by quadratures (see, for example, Goursat [8], ch. 29, Ex. 3). We make an application of this remark below.

It is clear that if $\psi(u)$ becomes infinite for some $0 < u < 1$ then $b(z) = 0$ for every $z > 0$. But since $y(0) = y(1) = 0$ it is possible that $\psi(u)$ may become infinite for $u = 0$ or 1 and still yield a nondegenerate $b(z)$. But in this case it is necessary that $\psi(u)$ not dominate $[2u(1 - u) \log \log 1/(u(1 - u))]^{-1}$ for u near 0 or 1.

We shall consider several examples.

Example 1. Let $\psi(u)$ be a constant over a set of intervals,

$$\psi(u) = q_k \geqq 0, \qquad u_k < u \leqq u_{k+1}, \qquad u_0 = 0, \qquad u_{n+1} = 1, \qquad k = 0, 1, \cdots n.$$

By choosing enough intervals, an arbitrary weight function can be approximated, in a manner of speaking.

It follows that the problem will be essentially solved if we can determine the functions $p_k(t_k, y; t, x)$ of Theorem 5.2. In this case the function $\xi(t)$ becomes, by (5.2),

$$\xi(t) = \frac{z}{\sqrt{q_k}}(1 + t), \qquad \frac{u_k}{1 - u_k} < t \leqq \frac{u_{k+1}}{1 - u_{k+1}},$$

and we must find the solution to equation (5.3) which satisfies the conditions (2) and (3) of Theorem 5.1.

As before we put $t_k = u_k/(1 - u_k)$, and it follows by a classical procedure of superposing an infinite system of sources and sinks along the line $t = t_k$ that we may get the Green's solution. In fact, let us put a source at $t = t_k$, $x = y_j$, of strength s_j, where

$$y_j = 2j \frac{z}{\sqrt{q_k}}(t_k + 1) + (-1)^j y,$$

$$s_j = (-1)^j \exp\left\{-2\frac{z^2}{q_k}(t_k + 1)j^2 - 2\frac{z}{\sqrt{q_k}}yj(-1)^j\right\}$$

for $j = 0, \pm 1, \pm 2, \cdots$. Then for $t_k < t \leqq t_{k+1}$ and $|y| < (z/\sqrt{q_k})(1 + t_k)$ we obtain

$$p_k(t_k, y; t, x) = \sum_{j=-\infty}^{\infty} \frac{s_j}{\sqrt{2\pi(t - t_k)}} e^{-\frac{1}{2}(x-y_j)^2/(t-t_k)}, \tag{5.7}$$

which may be directly verified by substitution to be a solution. It has been tacitly assumed that $q_k > 0$; if $q_k = 0$ we obtain only the term corresponding to $j = 0$ in the above solution, namely, the fundamental solution

$$p(t_k, y; t, x) = \frac{1}{\sqrt{2\pi(t - t_k)}} e^{-\frac{1}{2}(x-y)^2/2(t-t_k)}.$$

Now on putting

$$r_k = \min\left\{\frac{z}{\sqrt{q_k}}(1 + t_k), \frac{z}{\sqrt{q_{k-1}}}(1 + t_k)\right\}, \qquad k = 1, 2, \cdots, n,$$

and using the method outlined above, we obtain

$$g_n(t, x) = \int_{-r_n}^{r_n} \cdots \int_{-r_2}^{r_2} \int_{-r_1}^{r_1} p_0(0, 0; t_1, x_1) p_1(t_1, x_1; t_2, x_2) \cdots p_n(t_n x_n; t, x) \, dx_1 \, dx_2 \cdots dx_n$$

for $p_k(t_k, x_k; t_{k+1}, x_{k+1})$ as in (5.7), and finally as an "explicit" solution,

$$b_1(z) = \lim_{t \to \infty} \int_{|x| < \frac{z}{\sqrt{q_n}}(1+t)} g_n(t, x) \, dx.$$

The resulting function $b_1(z)$ is a multiply infinite sum of integrals of an n-variate Gaussian distribution over an n-dimensional rectangle.

We consider now the following special case of the above result

$$\psi(u) = \begin{cases} 1, & 0 \leqq a < u \leqq b \leqq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the test of the hypothesis is confined to detecting discrepancies over only a central portion of the interval [0, 1]. Using the preceding notation we have $n = 2$ and

$$u_0 = 0, \quad t_0 = 0, \quad q_0 = 0,$$

$$u_1 = a, \quad t_1 = \frac{a}{1-a}, \quad q_1 = 1,$$

$$u_2 = b, \quad t_2 = \frac{b}{1-b}, \quad q_2 = 0,$$

and hence

$$p(0, 0; t_1, x_1) = \frac{e^{-(x_1^2)/(2t_1)}}{\sqrt{2\pi t_1}},$$

$$p_1(t_1, x_1; t_2, x_2) = \sum_{j=-\infty}^{\infty} \frac{s_j}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{1}{2}(x_2 - y_j)^2/(t_2 - t_1)},$$

$$\text{(5.8)} \qquad \begin{cases} y_j = 2jz(t_1 + 1) + (-1)^j x_1, \\ s_j = (-1)^j \exp\{-2z^2(t_1 + 1)j^2 - 2zx(-1)^j\}, \end{cases}$$

$$p_2(t_2, x_2; t, x) = \frac{e^{-\frac{1}{2}(x - x_2)^2/(t - t_2)}}{\sqrt{2\pi(t - t_2)}}.$$

Thus, putting $b_1(z) = P(a, b, z)$,

$$P(a, b, z) = \lim_{t\to\infty} \int_{-\infty}^{\infty} \int_{-z(1+t_2)}^{z(1+t_2)} \int_{-z(1+t_1)}^{z(1+t_1)} p(0, 0; t_1, x_1)\, p_1(t_1, x_1; t_2, x_2) p_2(t_2, x_2; t, x)\, dx_1\, dx_2\, dx$$

$$= \int_{-z(1+t_2)}^{z(1+t_2)} \int_{-z(1+t_1)}^{z(1+t_1)} p(0, 0; t_1, x_1) p_1(t_1, x_1; t_2, x_2)\, dx_1\, dx_2$$

$$= \sum_{j=-\infty}^{\infty} \frac{s_j}{\sqrt{2\pi t_1(t_2 - t_1)}} \int_{-z(1+t_1)}^{z(1+t_1)} \int_{-z(1+t_2)}^{z(1+t_2)} \exp\left(-\frac{x_1^2}{2t_1} - \frac{(x_2 - y_j)^2}{2(t_2 - t_1)}\right) dx_2\, dx_1$$

for s_j and y_j as in (5.8).

The double integral is seen to be over a bivariate normal distribution, and if we let $n(x_1, x_2, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ be the normal bivariate density in x_1, x_2 with means μ_1, μ_2, variances σ_1^2, σ_2^2 and correlation ρ we obtain by rewriting the above integral

$$P(a, b, z) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2z^2j^2} \int_{-z(1+t_1)}^{z(1+t_1)} \int_{-z(1+t_2)}^{z(1+t_2)} n(x_1, x_2, \mu_1^{(j)}, \mu_2^{(j)}, \sigma_1^2, \sigma_2^2, \rho_j) dx_2 dx_1,$$

where

$$\mu_1^{(j)} = -2zj(-1)^j t_1, \quad \mu_2^{(j)} = 2zj, \quad \sigma_1^2 = t, \quad \sigma_2^2 = t_2, \quad \rho_j = (-1)^j \sqrt{\frac{t_1}{t_2}}.$$

A somewhat simpler way of writing this result is as follows. Let $M(u, v, \xi, \eta, \rho)$ be the volume under the normal bivariate surface with means zero and variances 1 and correlation ρ which is above the rectangle with vertices

$$x = u \pm \xi,$$
$$y = v \pm \eta.$$

Then, remembering that $t_1 = a/(1 - a)$, $t_2 = b/(1 - b)$ and $M(u, v, \xi, \eta, \rho) = M(-u, v, \xi, \eta, -\rho)$, we obtain after a simple transformation of the above integral

$$(5.9)\quad \begin{aligned} P(a, b, z) = \sum_{j=-\infty}^{\infty} (-1)e^{-2z^2j^2} \\ \cdot M\left(2jz\sqrt{\frac{a}{1-a}}, 2jz\sqrt{\frac{1-b}{b}}, \frac{z}{\sqrt{a(1-a)}}, \frac{z}{\sqrt{b(1-b)}}, -\sqrt{\frac{a(1-b)}{b(1-a)}}\right).\end{aligned}$$

There are tables available in which the function M is tabulated; see also Pólya [14]. Also, if either $a = 0$ or $b = 1$ then $\rho = 0$ and the function can be calculated with the ordinary univariate Gaussian tables. Putting $a = 0$, $b = 1$ simultaneously we obtain Kolmogorov's result

$$P(0, 1, z) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2j^2z^2},$$

which has been tabulated [18]. In the general case the convergence is very rapid and good results can be obtained by using a few central terms (in (5.9) the terms corresponding to $\pm j$ are clearly equal).

The formula (5.9) is in disagreement with a recent announcement (without proof) of Maniya [13]. Maniya's note appeared subsequent to a restricted paper by the authors.

By using the general formula above it is possible to get, for example, a weight function to test discrepancies over only the tails of the distribution, etc.

Example 2. We next investigate

$$\psi(u) = \begin{cases} \dfrac{1}{u(1-u)}, & 0 < a < u \leqq b < 1, \\ 0, & \text{otherwise,} \end{cases}$$

which is the weight function considered before with the W^2 test. By an earlier remark we must have $a > 0$ and $b < 1$, else absorption is certain and $b(z)$ is degenerate. The transformation (5.2) yields

$$b_2(z) = \Pr\left\{ |X(t)| < z\sqrt{t}, \frac{a}{1-a} < t < \frac{b}{1-b}\right\},$$

where $X(t)$ is the Wiener-Einstein process. Here it appears convenient to make another transformation. Let $u(t)$ be the Uhlenbeck process with correlation parameter β; that is, $u(t)$ is stationary Gaussian and Markovian with $E(u(s)u(t)) = \exp(-\beta|t-s|)$. Then from the known correspondence (cf. Doob [5])

$$X(t) = \sqrt{t}\, u\left(\frac{1}{2\beta}\log t\right)$$

we obtain

$$b_2(z) = \Pr\left\{ |u(t)| \leqq z, \frac{1}{2\beta}\log\frac{a}{1-a} < t < \frac{1}{2\beta}\log\frac{1}{1-b}\right\},$$

or since the process is strictly stationary

$$b_2(z) = \Pr\left\{ |u(t)| \leqq z, 0 < t < \frac{1}{2\beta}\log\frac{b(1-a)}{a(1-b)}\right\},$$

which is an absorption probability with a uniform barrier.

The function $b_2(z)$ is of some importance in the theories of communications and statistical equilibrium (cf. Bellman and Harris [1]), and may eventually be tabulated. It seems very difficult to give a complete analysis, but the following partial result is given without proof.

Let $\alpha = \frac{1}{2}\log(b(1-a))/(a(1-b))$ so that $b_2(z)$ is a function of α. Then it is possible to find the Laplace transform of $b_2(z)$ in the following form:

$$\int_0^\infty e^{-\lambda\alpha} b_2(z)\, d\alpha = \frac{1}{\lambda}\left\{1 - \sqrt{\frac{2}{\pi}}\frac{e^{\frac{1}{2}z^2}}{D_{-\lambda}(\sqrt{2}z) + D_{-\lambda}(-\sqrt{2}z)} \int_0^z e^{-\xi^2}\{D_{-\lambda}(\sqrt{2}\xi) + D_{-\lambda}(-\sqrt{2}\xi)\, d\xi\right\},$$

where $D_n(z)$ is the Weber function [23]. It seems very difficult to get even any qualitative information from this formula.

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