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A COMPARATIVE EVALUATION OF THE ESTIMATORS OF THE LOG PEARSON TYPE (LP) 3 DISTRIBUTION

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ABSTRACT

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The log Pearson type 3 (LP3) distribution, recommended by the U.S. Water Resources Council (USWRC) in 1967, and subsequently updated in 1975, 1977 and 1981 as the base method of flood frequency analysis in the United States, has been widely used in many parts of the world. However, the estimation procedure for the LP3 distribution recommended by the USWRC has been shown by many investigators to perform rather poorly. In this study several estimation methods for the LP3 distribution, some quite recent, are investigated. The performance of the various methods is then compared via Monte Carlo simulation with the objective of identifying the most robust estimator among them. It is found that besides the USWRC method, the methods based on maximum likelihood and entropy perform poorly. The method based on the moments in real space, and the relatively recent so-called method of mixed moments perform markedly superior to other methods in terms of both resistance and efficiency of estimation or robustness. Therefore, the USWRC guidelines are in need of revision.

INTRODUCTION

The objective of flood frequency analysis is to obtain an estimate of the T-year flood quantile at one or more locations in a river system. The quantile estimates at any site are subject to considerable inherent variability. Several factors dictate the extent of variability, chief among them being the sample size, n, the recurrence interval of estimate, and the estimator utilized. The choice of estimator becomes crucial since competing estimators yield estimates markedly differing in bias and mean square error – the commonly used statistical measures of performance.

Considerable uncertainty exists about the form of the underlying population distribution of floods at any site. Owing to the vast hydrogeological variations possible, it is reasoned that the population distribution may have remarkably wide range of forms for various sites. This is compounded by the limited amount of data that is typically available to estimate a model for the flood distribution. These factors dictate that robust estimators be searched which are efficient and resistant over wide range of populations characteristic of real-world floods (Kuczera, 1982a, b).

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The structure of the estimator depends upon the model selected (e.g., log normal, log Pearson, etc.) and the method of estimation used (e.g., method of moment, maximum likelihood, etc.). Several models ranging from the two-parameter extreme value type 1 (EV1) to the five-parameter Wakeby distribution have been used for flood frequency analysis. The flexibility of a model increases as the number of parameters goes up. However, the variability of estimation also goes up for an estimator that is based on distribution having more parameters. Thus, while an estimator based on EV1 might considerably underestimate the quantiles with low variability, an estimator based on Wakeby might prove to be so unstable (variable) as to be suitable for only regional analysis. Since one of the criteria for a robust estimator is resistance (stability) to population fluctuations, it seems reasonable to expect that an estimator based on a distribution with fewer parameters would be more stable than one based on a distribution with more parameters. The three-parameter models seem likely candidates to provide a good balance between the opposing factors of flexibility and variability, as they are flexible enough to accommodate the wide population distribution forms that might reasonably be expected, and yet not exhibit undue deterioration in variability. In this paper we examine the estimators based on one such three-parameter model – the log Pearson type 3 (LP3) distribution. The LP3 is a popular distribution which has been widely used for flood frequency analysis.

Several studies have been reported in the literature comparing the performance of "at-site", "at-site/regional", "regional only" estimators or combinations thereof. In a site-specific framework, a parent distribution representative of real flood experience (Landwehr et al., 1980) is selected and several competing estimators based on specified model distributions are used to estimate quantiles from the samples drawn from the parent distribution. Kuczera (1982b) considered the performance of the following estimators on four Wakeby (WAK) parent distributions: (1) normal distribution with maximum likelihood estimation (MLE); (2) two-parameter log normal (LN2) distribution with MLE, and robust method of moment (MOM); (3) LP3 distribution with MOM applied to log-transformed data (referred subsequently as indirect method of moment (MMI)); (4) extreme value type 1 (EV1) distribution with MLE, probability weighted moment (PWM) and MOM; (5) log-EV1 distribution with MOM; and (6) WAK with PWM. He found that the LP3 and log-EV1 distributions, because of their flexible shapes, were better able to mimick the parent distribution shapes, i.e., they exhibited small bias. However, this was negated by their poor standard errors, so that they performed less efficiently than the LN2 and EV1 distributions. But interestingly enough, the poor performance of the LP3 and log-EV1 estimators was not solely due to their three parameter structure, since the four parameter Wakeby distribution with PWM performed better than either of them. This strongly suggests that the estimation method plays an important role. In a more recent study, Wallis and Wood (1985) compared the performance of the general extreme value (GEV) distribution with PWM; the WAK distribution with PWM; and the log-Pearson type 3 (LP3) distribution with MMI conditioned on GEV, WAK and LP3 populations.

While many more studies have been reported on comparison of the performance of quantile estimators, the ones mentioned above have LP3 with MMI figuring as one of the competing estimators. The general conclusion of all these studies seems to be that the LP3 distribution with MMI depicts poorer performance. This comes as little surprise since the LP3 distribution being a three parameter model, its MMI quantile estimator depends on the sample skewness estimate – a statistic with a significant downward bias (Wallis et al., 1974), algebraic bounds (Kirby, 1974), and large sampling variance. However, the LP3 distribution has been found to be a flexible distribution, theoretically capable of modeling the large skewness and kurtosis behavior of real flood series (Bobee, 1975; Rao, 1980a; Hoshi and Burges, 1981). It seems plausible, therefore, that the poor performance of the LP3-based MMI estimator stems not so much from the distribution as from the estimation method. In light of these observations, it is natural to search for better quantile estimators of the LP3 distribution.

Much interest has been generated in the LP3 distribution since it was first recommended by the U.S. Water Resources Council (USWRC, 1967), and subsequently updated in 1975, 1977 and 1981 as the base method of flood frequency analysis in the United States. Bobee (1975) studied the theoretical properties of the LP3 distribution and suggested an estimation method based on the moments of the real data whereas the USWRC has advocated an estimation method based on the moments of the log-transformed data. Condie (1977) studied the MLE estimates of the LP3 distribution and derived analytical results for calculating the asymptotic standard error of the MLE quantile estimator. Nozdryn-Plotnicki and Watt (1979) carried out sampling experiments to compare the performance of the three estimation methods over the LP3 parameter space representative of Canadian flood data. Rao (1980b, 1983) proposed a new method called the method of mixed moments (MIX) which obviated the need to use the sample estimates of the skewness coefficients in estimating the parameters and quantiles via sample moment estimates. For the parameter space considered by him, MIX has performed well in comparison with other methods. Singh and Singh (1985) proposed another new estimation method based on the concept of entropy.

The available estimators of the LP3 model were examined in detail and suitable algorithms devised for each. It is noted here that considerable confusion exists in literature about the solution strategy of some of these estimation procedures. In particular, a comprehensive algorithm was devised for affecting the MLE solution. Also a simplified procedure for finding MIX estimates was obtained. Next, in a comprehensive within-distribution study, the estimators were tested on Monte Carlo synthetic samples simulating a wide range of population characteristics. This study reports the significant findings of our work along with the statistical performance of each estimator, and the identification of the most robust estimator(s).

THE LP3 DISTRIBUTION

Let $Y = \ln X$ be a Pearson type 3 variate. The probability density function of Y is given by:

$$g(y) = \frac{1}{|a|\Gamma(b)}\left[\frac{y-c}{a}\right]^{b-1} \exp\left[-\left(\frac{y-c}{a}\right)\right] \quad (1)$$

The quantity X, by definition, is log-Pearson 3 variate, and its probability density function, easily derived from eqn. (1), is given by:

$$f(x) = \frac{1}{|a|x\Gamma(b)}\left[\frac{\ln x - c}{a}\right]^{b-1} \exp\left[-\frac{\ln x - c}{a}\right] \quad (2)$$

where a, b and c are the scale, shape, and location parameters respectively. The parameter b is positive and $\Gamma(b)$ denotes the gamma function.

The mean, variance, and skewness coefficient of Y are given by:

$$\text{Mean: } \mu_y = c + ab \quad (3)$$

$$\text{Variance: } \sigma_y^2 = ba^2 \quad (4)$$

$$\text{Skew: } \gamma_y = \frac{|a|}{a}\frac{2}{b^{1/2}} \quad (5)$$

The moments of X are given by:

$$\mu_r' = \frac{\exp(rc)}{(1-ra)^b}, \quad 1 - ra > 0, \quad r = 1, 2, 3 \quad (6)$$

where μ_r' denotes the rth moment of X about the origin.

From eqn. (6) the mean, variance, coefficient of variation (CV), skewness coefficient (skew), and kurtosis of X are given by:

$$\text{Mean: } \mu = \frac{\exp(c)}{(1-a)^b} \quad (7)$$

$$\text{Variance: } \sigma_x^2 = \exp(2c)\, A \quad (8)$$

$$\text{Coefficient of variation: } \beta = (1-a)^b A^{1/2} \quad (9)$$

$$\text{Skewness coefficient: } \gamma = \left[\frac{1}{(1-3a)^b} - \frac{3}{(1-a)^b(1-2a)^b} + \frac{2}{(1-a)^{3b}}\right]/A^{3/2} \quad (10)$$

$$\text{Kurtosis: } \lambda = \left[\frac{1}{(1-4a)^b} - \frac{4}{(1-a)^b(1-3a)^b} + \frac{6}{(1-a)^{2b}(1-2a)^b} - \frac{3}{(1-a)^{4b}}\right]A^{-2} \quad (11)$$

where:

$$A = \left[\frac{1}{(1-2a)^b} - \frac{1}{(1-a)^{2b}}\right]$$

We note that the coefficients of variation, skewness, and kurtosis in eqns. (9)–(11) are independent of the location parameter c.

Consider eqn. (5). If $a > 0$, then $\gamma_y > 0$ implying Y is positively skewed and $c < Y < +\infty$. In this case, X is also positively skewed (Rao, 1980a), and $\exp(c) < X < +\infty$. If $a < 0$, then $\gamma_y < 0$ implying that Y is negatively skewed and $-\infty < Y < c$. In this case, X is either positively or negatively skewed depending upon the values of the parameters a and b, and $-\infty < X < \exp(c)$. For this case, the density $f(x) = 0$, may be arbitrarily defined as zero (Rao, 1980a).

The overall geometric shape of the LP3 distribution is governed by the parameters a and b (Rao, 1980a; Bobee, 1975). The probability density function is capable of assuming diverse shapes such as reverse-J, U, J, and of course, the unimodal (skewed) bell shape. Hoshi and Burges (1981a) point out that if $\gamma < \beta^3 + 3\beta$ then $a < 0, 0 < x < \exp(c), \lambda_{LP3} < \lambda_{LN3}$, and vice versa, where λ_{LP3} is the coefficient of kurtosis for the LP3 distribution eqn. (11), and λ_{LN3} is the coefficient of kurtosis for the three-parameter log normal (LN3) distribution. The LP3 distribution degenerates to the log normal distribution when the parameters a and b become zero and infinity respectively (or equivalently, when $\gamma = \beta^3 + 3\beta$ and $\gamma_y = 0$).

METHODS OF PARAMETER ESTIMATION

Method of moments (direct) — MMD

This method, proposed by Bobee (1975), uses the sample estimates of moments of the untransformed (real) data. Using eqn. (6), we can write:

$$\ln \mu_1' = c - b \ln(1 - a) \quad (12)$$

$$\ln \mu_2' = 2c - b \ln(1 - 2a) \quad (13)$$

$$\ln \mu_3' = 3c - b \ln(1 - 3a) \quad (14)$$

Equations (12)–(14) can be rearranged to give:

$$\frac{\ln \mu_3' - 3 \ln \mu_1'}{\ln \mu_2' - 2 \ln \mu_1'} = \frac{3 \ln(1-a) - \ln(1-3a)}{2 \ln(1-a) - \ln(1-2a)} \ (= B \text{ say}) \quad (15)$$

For the sample under consideration, $B = (\ln \mu_3' - 3 \ln \mu_1')/(\ln \mu_2' - 2 \ln \mu_1')$ can be estimated from the sample estimates of the first three moments μ_1', μ_2', μ_3' about the origin $(= n^{-1} \sum_{i=1}^{n} x_i^j, j = 1, 2, 3)$. The right-hand side of eqn. (15), which is a function of the parameter a only [say $B(a)$], reveals that $a < 1/3$. In the

TABLE 1

Estimating scale parameter a; direct method of moments (MMD)

a	b	a	b	a	b
−4000.00000	2.03784	−0.00050	2.99900	0.11111	3.30930
−3000.00000	2.03932	−0.00040	2.99920	0.12500	3.36624
−2000.00000	2.04162	−0.00030	2.99940	0.13333	3.40343
−1000.00000	2.04623	−0.00020	2.99960	0.14006	3.43528
−500.00000	2.05195	−0.00010	2.99980	0.14045	3.43720
−250.00000	2.05923	−0.00005	2.99990	0.14085	3.43914
−200.00000	2.06201	−0.00001	2.99998	0.14124	3.44109
−125.00000	2.06873	0.00000	3.00000	0.14164	3.44306
−100.00000	2.07242	0.00001	3.00002	0.14205	3.44505
−50.00000	2.08654	0.00005	3.00010	0.14225	3.44605
−25.00000	2.10634	0.00010	3.00020	0.14265	3.44807
−12.50000	2.13498	0.00020	3.00040	0.14306	3.45010
−10.00000	2.14684	0.00030	3.00060	0.14327	3.45113
−5.00000	2.19521	0.00040	3.00080	0.14368	3.45319
−2.50000	2.26716	0.00050	3.00100	0.14409	3.45527
−2.00000	2.29663	0.00060	3.00120	0.14443	3.45695
−1.25000	2.36969	0.00070	3.00140	0.14455	3.45759
−1.00000	2.40942	0.00080	3.00160	0.14472	3.45843
−0.50000	2.54794	0.00090	3.00180	0.14684	3.46929
−0.40000	2.59470	0.00100	3.00200	0.15129	3.49264
−0.33333	2.63252	0.00125	3.00251	0.15601	3.51846
−0.25000	2.69009	0.00167	3.00335	0.16103	3.54716
−0.20000	2.73193	0.00200	3.00402	0.16367	3.56275
−0.12500	2.80904	0.00250	3.00503	0.16920	3.59680
−0.10000	2.83972	0.00333	3.00672	0.17513	3.63528
−0.06667	2.88561	0.00400	3.00808	0.18149	3.67915
−0.05000	2.91106	0.00500	3.01013	0.18484	3.70348
−0.04000	2.92725	0.00667	3.01356	0.19194	3.75786
−0.03333	2.93845	0.00800	3.01632	0.19569	3.78839
−0.02500	2.95293	0.01000	3.02051	0.20367	3.85765
−0.02000	2.96190	0.01111	3.02286	0.20790	3.89715
−0.01667	2.96800	0.01250	3.02581	0.21692	3.98852
−0.01250	2.97576	0.01400	3.02878	0.22173	4.04177
−0.01111	2.97838	0.01667	3.03478	0.22676	4.10126
−0.01000	2.98049	0.02000	3.04211	0.23202	4.16819
−0.00800	2.98431	0.02222	3.04706	0.23753	4.24411
−0.00500	2.99012	0.02500	3.05334	0.24331	4.33101
−0.00400	2.99208	0.02941	3.06351	0.24938	4.43154
−0.00333	2.99339	0.03030	3.06559	0.25575	4.54934
−0.00250	2.99503	0.03333	3.07275	0.26247	4.68948
−0.00200	2.99602	0.04000	3.08893	0.26954	4.85935
−0.00125	2.99751	0.05000	3.11437	0.27701	5.07016
−0.00100	2.99800	0.06667	3.16024	0.28490	5.33998
−0.00090	2.99820	0.07692	3.19087	0.29326	5.70016
−0.00080	2.99840	0.08333	3.21106	0.30211	6.21160
−0.00070	2.99860	0.09091	3.23603	0.31153	7.01497
−0.00060	2.99880	0.10000	3.26772	0.32154	8.56194

$B = [3\ln(1-a) - \ln(1-3a)]/[2\ln(1-a) - \ln(1-2a)]$

limit, $B(a)$ approaches ∞, 3, and 2 as a approaches 1/3, 0, and $-\infty$ respectively. It should be possible to approximate the $B(a)$ versus a relation by a series of polynomials, as for example in Kite (1977). Then a good approximation of sample estimate of a could directly be found from the sample estimated value of B and should be good enough for most fitting problems. However, for purposes of simulation, a large number of $[a-B(a)]$ points (Table 1) were generated in the region $a < 1/3$ (Bobee, 1975). Subsequently, a sample estimate of a was interpolated corresponding to the sample estimated B value from the generated $a-B(a)$ points, and refined using the Newton-Raphson method applied to eqn. (15). With the interpolated value of a being a very good starting solution, the iterative scheme quickly converged to the true solution to any desired degree of significant digit accuracy. The parameters b and c were then estimated using eqns. (12) and (13).

Method of moments (indirect) — MMI1 and MMI2

This is basically the method advocated by the U.S. Water Resources Council. It is the method of moments applied to the log-transformed data. The method utilizes eqns. (3)–(5) for estimating the parameters. Details of the method can be found in U.S. Water Resources Council's Bulletin Nos. 15, 17A and 17B, Rao (1980b), amongst others. Two variations of MMI were tested in simulation studies here. They essentially differed in the sample skewness estimator used on the log-transformed data:

$$g_y = \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} \frac{(y_i - \bar{y})^3}{S_y^3} \tag{16}$$

$$g_y' = \left(1 + \frac{8.5}{n}\right) g_y \tag{17}$$

where n is the sample size, and $\bar{y}$ and S_y are the sample mean and standard deviation respectively of the log-transformed data.

Method of mixed moments — MIX

Rao (1980b, 1983) proposed this method for LP3 with the objective of obviating the use of the sample skewness coefficient in parameter estimation. He tried various combinations of mixing the first two moments of the untransformed and log-transformed samples and found one particular combination to be preferable on the basis of sampling properties. The method, referred to here as MIX, is outlined in the Appendix, which also details a simplified procedure to obtain MIX parameter estimates.

Method of maximum likelihood estimation — MLE

The MLE method in the context of LP3 distribution (or equivalently, Pearson (P) 3 with log-transformed data) has been investigated by Matalas et al. (1973), Condie (1977), Nozdryn-Plotnicki and Watt (1979), and Rao (1986), among others. The procedures proposed by all the investigators, except Rao (1986), essentially involve searching for the solution within a limited range of parameter c, for example, ($\ln x_{\min} - 0.1$, $\ln x_{\min} - 50$) for positive g_y, and ($\ln x_{\max} + 0.01$, $\ln x_{\max} + 50$) for negative g_y, where $x_{\min}$, $x_{\max}$ are the minimum and maximum values of the untransformed sample, and g_y is the estimated skewness coefficient of the log-transformed sample. A thorough investigation was made by the authors into the MLE procedure for LP3 (Arora and Singh, 1987) and an algorithm proposed to effect the solution. The procedure that evolved is briefly discussed below for the sake of continuity.

The log-likelihood function can be written as:

$$\begin{aligned} LL &= \sum_{i=1}^{n} \ln f(x_i;\, a, b, c) \\ &= -n \ln |a| - \sum \ln x - n \ln \Gamma_b + (b-1) \sum \ln [(\ln x - c)/a] \\ &\quad - (1/a) \sum (\ln x - c) \end{aligned} \tag{18}$$

and the summations are over n sample values wherever not specified.

The objective of the method is to maximize the likelihood function, or equivalently, to maximize the log-likelihood function eqn. (18). Thus, the following parameter estimation equations result:

$$\frac{\partial(LL)}{\partial a} = -nab + \sum (\ln x - c) = 0 \tag{19}$$

$$\frac{\partial(LL)}{\partial b} = -n\psi(b) + \sum \ln \left[\frac{(\ln x - c)}{a}\right] = 0 \tag{20}$$

$$\frac{\partial(LL)}{\partial c} = \frac{n}{a} - (b-1) \sum \frac{1}{(\ln x - c)} = 0 \tag{21}$$

Equations (19) and (21) can be rearranged (Rao, 1986) to give:

$$a = \frac{s_1}{nb} \tag{22}$$

$$b = \frac{s_1 s_2}{(s_1 s_2 - n^2)} \tag{23}$$

where $s_1 = \sum (\ln x - c)$ and $s_2 = \sum 1/(\ln x - c)$.

For a specified value of c, parameters b and a can be explicitly found from eqns. (23) and (22) respectively. Substitution of these a, b and c values in eqn. (20) yields $\partial(LL)/\partial b = R$.

The following algorithm was employed to find the MLE parameter estimates in simulation studies. Let the population $c > 0$.

Step (1): Search for c in the range $[\ln(x_{\max}/\bar{x}) + 0.01, \ln(x_{\max}/\bar{x}) + 50]$. If it exists, find it by regula-falsi method, and calculate the corresponding *LL* (18). If *LL* is a local maximum, STOP: *c* is the MLE solution. Else, go to step (2).

Step (2): Search for *c* in the range $[\ln(x_{max}/\bar{x}) - 0.01, \ln(x_{max}/\bar{x}) - 50]$. If it exists and is a local maximum, STOP. MLE solution is found. Else, go to step (3).

Step (3): Find $c_2 \rightarrow +\infty$, corresponding to $|R| \leqslant 10^{-9}$. This is the MLE solution.

For population $c < 0$, a complementary procedure was followed.

Method of entropy — ENT

Singh and Singh (1985) used the concept of entropy to derive a new set of estimation equations for the Pearson type 3 distribution. The parameter estimation equations, with theoretical expectations replaced by sample moments are:

$$\sum(\ln x - c) = nab \tag{24}$$

$$S_y^2 = a^2 b \tag{25}$$

$$\sum \ln\left[\frac{\ln x - c}{a}\right] = n\psi(b) \tag{26}$$

where S_y^2 is the sample variance estimate from the log-transformed data. It is interesting to note that two equations of this system, namely eqns. (24) and (26), are exactly identical to two MLE equations, namely eqns. (19) and (20). The third eqn. (25) appears to be simpler than the corresponding eqn. (21). In fact, eqn. (25) is the same as eqn. (4) which is also used in MMI.

The system eqns. (24)–(26) offers similar computational problems as MLE. Equation (26) requires that $(\ln x_i - c)$ and a should have the same signs. Therefore, a is positive, if $c < \ln x_i$, that is, c is a lower bound, and vice versa. Equation (24) [or (25)] allows for $b > 0$, whereas eqn. (21) of MLE requires $b > 1$. Therefore, the ENT system of equations is less restrictive than MLE system.

Equations (24)–(26) have multiple roots. After rearranging, the equations can be written as:

$$a = \pm S_y/\sqrt{b} \tag{27}$$

$$c = \bar{y} \pm S_y\sqrt{b} \tag{28}$$

Hence, for every value of b, we have two possible values of a and c. The ENT solution was obtained by following similar procedure as in the case of MLE.

QUANTILE ESTIMATION

Let:

$$W = \frac{\ln X - c}{a} \tag{29}$$

W is the standard gamma variate with shape parameter b (and scale parameter 1) having the probability density function:

$$g(w) = \frac{\exp(-w)\, w^{b-1}}{\Gamma(b)}, \quad w \geq 0 \qquad (30)$$

The cumulative distribution function $F(x)$ of the LP3 variate X can be expressed as:

$$\begin{aligned} F(x) &= \text{Prob}\,(X \leq x) \\ &= \text{Prob}\left[\frac{\ln X - c}{a} \leq \frac{\ln x - c}{a}\right], \quad a > 0 \\ &= \text{Prob}\left[\ln \frac{X - c}{a} \geq \frac{\ln x - c}{a}\right], \quad a < 0 \end{aligned} \qquad (31)$$

That is:

$$\begin{aligned} F(x) &= G(w_x), \quad a > 0 \\ &= 1 - G(w_x), \quad a < 0 \end{aligned} \qquad (32)$$

where $G(w_x)$ is the cumulative distribution function of the standard Gamma variate W defined as:

$$G(w_x) = \int_0^{w_x} g(w)\, \mathrm{d}w, \qquad w_x = \frac{\ln x - c}{a} \qquad (33)$$

Therefore, the T-year return period quantile, x_T, can be obtained from:

$$\ln x_T = c + a\, w_T \qquad (34)$$

where w_T is a standard Gamma quantile corresponding to nonexceedance probability $[G(w_T)]$ of $(1 - 1/T)$ for $a > 0$, or $1/T$ for $a < 0$. w_T can be written as:

$$w_T = b^{1/2}(K_T + b^{1/2}) \qquad (35)$$

where K_T is the frequency factor corresponding to the same probability level as w_T. Using eqn. (35), we can write eqn. (34) as:

$$\begin{aligned} \ln x_T &= c + ab + ab^{1/2} K_T \\ &= \mu_y + \sigma_y K_T \end{aligned} \qquad (36)$$

K_T depends on the probability level, the corresponding gamma quantile, w_T, shape parameter b, and sign of parameter a (or equivalently γ_y). Its values are tabulated in U.S. Water Resources Council Bulletin 17 for $\gamma_y = -9.0$ to 9.0 for a wide range of exceedance probability levels. The values of K_T were interpolated from this table and used in eqn. (36) to obtain the quantile estimate x_T.

EXPERIMENTAL DESIGN

To assess the performance of various methods of estimation outlined above, Monte Carlo sampling experiments were performed. Annual flood data generally lie in the area of the β–γ diagram delineated by $0.3 < \beta < 0.8$ and γ upward of 1 (Landwehr et al., 1978; Wallis and Wood, 1985; Rossi et al., 1986). Based on this consideration, five cases of LP3 population, representative of the real flood data were selected for Monte Carlo experiments. These cases are listed in Table 2. It is noted here that $\lambda_1 < \lambda_{LN3} < \lambda_5 < \lambda_2 < \lambda_4 < \lambda_3$ where the subscripts of λ refer to the LP3 population case.

For each of the population cases listed in Table 2, 1000 random samples of size 10, 20, 30, 50 and 75 were generated, and in parameters and quantiles were estimated from the methods of estimation outlined earlier. The 1000 estimated values of estimated parameters and quantiles for each sample size and population case were used to approximate the values of the following performance indices for that case:

Standardized bias: $$BIAS = \frac{E(\hat{x}) - x}{x} \tag{37}$$

Standard error, $$SE = \frac{\sigma(\hat{x})}{x} \tag{38}$$

Root mean square error, $$RMSE = \frac{E[(\hat{x} - x)^2]^{1/2}}{x} \tag{39}$$

where $\hat{x}$ is an estimate (parameter or quantile) of x, $E(\cdot)$ denotes statistical expectation, and $\sigma(\cdot)$ denotes standard deviation of the respective random variable. $E(\hat{x})$ and $\sigma(\hat{x})$ were calculated as:

$$E(\hat{x}) = \frac{\sum \hat{x}}{N} \tag{40}$$

TABLE 2

($\mu = 1$): LP3 population cases considered in sampling experiments

LP3 population	CV (β)	Skew (γ)	Parameter a	Parameter b	Parameter c	γ_y
Case 1	0.5	1	−0.11832	19.82269	2.216713	−0.45
Case 2	0.5	3	0.127683	10.30311	−1.407434	0.62
Case 3	0.5	5	0.205678	3.215257	−0.740366	1.12
Case 4	0.3	3	0.150978	2.681889	−0.438946	1.22
Case 5	0.7	3	0.059798	98.38009	−6.066213	0.20

$$\sigma(\hat{x}) = \left[\frac{1}{N-1}\sum\{\hat{x} - E(\hat{x})\}^2\right]^{1/2} \tag{41}$$

where the summations are over N estimates $\hat{x}$ of x, N being the number of random samples used in estimation (= 1000 here).

It is easy to write:

$$RMSE = \left[\frac{N-1}{N} SE^2 + BIAS^2\right]^{1/2} \tag{42}$$

Due to the limited number of random samples used, the results are not expected to reproduce the true values of *BIAS*, *SE* and *RMSE*, but they do provide a means of comparing the performance of various estimation methods.

Equation (29) was used to facilitate the generation of LP3 random numbers. To start with, standard Gamma numbers, W_R, were generated through Gamma generator GGAMR (IMSL, 1981). Then the corresponding LP3 numbers, X_R, were obtained as

$$X_R = \exp(a\, W_R + c) \tag{43}$$

Robustness

The Monte Carlo experiments outlined above resulted in a large number of tables depicting the statistical properties of the estimators. However, due to limitations of space, only a small, representative portion of those results is presented here. Tables 3, 4 and 5 show the *BIAS*, *SE* and *RMSE* of the 200-year quantile estimators for sample sizes 10, 30 and 75. Our objective here is to identify robust estimator(s) based on the results of the Monte Carlo study. Kuczera (1982a, b) defined a robust estimator as one that is resistant and efficient over a wide range of population fluctuations. If an estimator performs steadily without undue deterioration in *RMSE* and *BIAS*, it can be expected to perform better than other competitive estimators under population conditions different from those on which the conclusions are based. Two criteria for identifying a resistant estimator (Kuczera, 1982b) are mini–max and minimum average *RMSE*. According to the mini–max criteria, the preferred estimator is the one whose maximum *RMSE* for the five population cases is minimum. The minimum average criterion is to select the estimator whose *RMSE* average over the five cases is minimum. Table 6 reports the maximum and average *RMSE* for each estimator for selected sample sizes and return period. There are wide differences in the *RMSE* performance of estimators with the percent difference between the best and worst being as much as 425% for sample size 10. Either MIX or MMD provides the most favorable *RMSE* values in all four columns of Table 6.

With reference to Table 6, the MIX estimator is superior on the basis of the minimum-average *RMSE* criteria, and comparable to MMD on the basis of

TABLE 3

Bias of 200-yr quantile estimators

Method	Sample size	LP3 population case (Table 2)				
		1	2	3	4	5
MMD	10	−0.158	−0.247	−0.280	−0.199	−0.291
MMI1	10	0.107	0.072	0.054	−0.024	0.172
MMI2	10	0.090	0.402	0.474	0.184	0.554
MIX	10	0.015	−0.108	−0.156	−0.080	−0.134
MLE	10	0.161	−0.166	−0.274	−0.214	−0.073
ENT	10	−0.179	0.268	0.144	−0.009	0.678
MMD	30	−0.066	−0.156	−0.167	−0.117	−0.167
MMI1	30	0.051	0.009	0.008	−0.016	0.032
MMI2	30	0.013	0.082	0.116	0.053	0.081
MIX	30	0.006	−0.080	−0.109	−0.062	−0.094
MLE	30	0.029	0.068	−0.009	−0.067	0.113
ENT	30	−0.084	0.080	−0.006	−0.049	0.221
MMD	75	−0.030	−0.083	−0.100	−0.065	−0.087
MMI1	75	0.026	0.012	0.013	0.001	0.024
MMI2	75	0.005	0.040	0.055	0.030	0.040
MIX	75	0.003	−0.042	−0.063	−0.031	−0.044
MLE	75	−0.005	0.048	0.053	−0.013	0.034
ENT	75	−0.034	0.040	−0.015	−0.047	0.115

TABLE 4

Standard error of 200-yr quantile estimators

Method	Sample size	LP3 population case (Table 2)				
		1	2	3	4	5
MMD	10	0.243	0.361	0.397	0.290	0.395
MMI1	10	0.438	0.878	1.033	0.537	1.127
MMI2	10	0.653	0.825	2.160	0.916	2.522
MIX	10	0.294	0.422	0.465	0.334	0.476
MLE	10	0.449	0.386	0.342	0.252	0.516
ENT	10	0.267	0.861	1.154	0.445	1.490
MMD	30	0.172	0.281	0.322	0.227	0.313
MMI1	30	0.233	0.398	0.480	0.314	0.424
MMI2	30	0.262	0.500	0.616	0.383	0.522
MIX	30	0.187	0.269	0.319	0.240	0.317
MLE	30	0.335	0.427	0.379	0.234	0.630
ENT	30	0.156	0.358	0.342	0.216	0.429
MMD	75	0.120	0.225	0.263	0.173	0.265
MMI1	75	0.151	0.258	0.309	0.204	0.282
MMI2	75	0.160	0.283	0.343	0.222	0.309
MIX	75	0.125	0.222	0.245	0.179	0.248
MLE	75	0.154	0.246	0.250	0.159	0.289
ENT	75	0.112	0.221	0.229	0.152	0.253

TABLE 5

Root mean square error of 200-yr quantile estimators

Method	Sample size	LP3 population case (Table 2)				
		1	2	3	4	5
MMD	10	0.290	0.438	0.486	0.351	0.491
MMI1	10	0.451	0.880	0.034	0.537	1.140
MMI2	10	0.658	0.868	0.210	0.934	2.581
MIX	10	0.294	0.436	0.490	0.334	0.494
MLE	10	0.477	0.420	0.438	0.330	0.520
ENT	10	0.321	0.901	0.162	0.445	1.636
MMD	30	0.184	0.316	0.363	0.255	0.355
MMI1	30	0.238	0.398	0.480	0.314	0.425
MMI2	30	0.262	0.507	0.626	0.386	0.528
MIX	30	0.187	0.300	0.337	0.248	0.330
MLE	30	0.336	0.432	0.378	0.244	0.639
ENT	30	0.177	0.367	0.342	0.221	0.482
MMD	75	0.123	0.240	0.282	0.185	0.279
MMI1	75	0.153	0.258	0.310	0.204	0.283
MMI2	75	0.160	0.286	0.347	0.224	0.311
MIX	75	0.125	0.226	0.253	0.181	0.252
MLE	75	0.154	0.250	0.255	0.160	0.291
ENT	75	0.117	0.225	0.230	0.159	0.278

TABLE 6

Summary of *RMSE* performance of 200-yr quantile estimators

Estimator	Maximum *RMSE*		Average *RMSE*	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$
MMD	0.491*	0.363	0.411*	0.295
MMI1	1.140	0.480	0.808	0.371
MMI2	2.581*	0.626	1.650*	0.462*
MIX	0.494	0.337*	0.412	0.280*
MLE	0.520	0.639*	0.437	0.406
ENT	1.636	0.482	0.893	0.318
Maximum % diff.	425	90	300	65

* Highest and lowest values.

TABLE 7

Summary of *BIAS* performance of 200-yr quantile estimators

Estimator	Maximum *BIAS*		Average absolute + * *BIAS*	
	$n = 10$	$n = 30$	$n = 10$	$n = 30$
MMD	−0.291	−0.167	−0.235	−0.133
MMI1	0.172	0.032	0.086	0.023
MMI2	0.554	0.116	0.341	0.069
MIX	−0.156	−0.109	−0.099	−0.070
MLE	−0.274	0.113	−0.178	0.057
ENT	0.678	0.221	0.256	0.088

* The sign in front of the values shows the dominant tendency of the estimator.

mini–max *RMSE* criteria. Hence, MIX is expected to be the most resistant estimator. Nevertheless, MMD performs comparably. MIX and MMD perform markedly superior to other methods. The method proposed by the U.S. Water Resources Council (MMI1) performs poorly, as do MLE and ENT. Taking into consideration the poor performance of MLE and ENT and the tremendous amount of CPU time required by the extensive search routines, there should be no doubt that MLE and ENT are inferior methods for LP3 distribution.

To see the performance of estimators in terms of *BIAS*, Table 7, similar to Table 6, was prepared. Interestingly enough, the superior *RMSE* performance of MIX in comparison to MMD is not deteriorated by *BIAS*. MIX yields considerably less *BIAS* than MMD and is clearly superior to MMD in terms of both mini–max *BIAS* and minimum average *BIAS* criteria. Trends shown in Tables 6 and 7 are typical, and similar results were observed for other sample sizes and return periods.

RESULTS AND DISCUSSION

In general, unusually high *BIAS*, *SE* and *RMSE* were observed for parameter estimators $\hat{a}$, $\hat{b}$ and $\hat{c}$ of all methods. However, the intercorrelation among the parameter estimates was such that reasonable quantile estimates were obtained.

MMD and MIX mostly underestimated the quantiles, especially for $T > 25$, as evident from their negative bias (Table 3). MIX consistently produced smaller bias than MMD, and the difference became more pronounced at higher return periods. MMI1 and MMI2 mostly overestimated the quantiles (positive bias). Such trends were not discernible for MLE and ENT. MMI1 mostly produced smaller absolute bias estimates than MIX.

In terms of standard error (Table 4), MMI1 and MMI2 consistently produced a higher standard error than other methods, especially MMD and MIX. MMI2 faired worse than MMI1. MLE and ENT seemed susceptible to smaller sample sizes, and in general produced a higher standard error than other methods at such sample sizes. MIX and MMD depicted remarkable stability even at smaller sample sizes when some of the other methods showed a deterioration in standard error. In general, MIX and MMD outperformed other estimators in terms of standard error for all population cases.

As compared to other estimators, MMI1 and MMI2 performed poorly in terms of root mean square error (Table 5). While MLE and ENT did perform well for some population cases and sample sizes, they depicted large deteriorations in root mean square error statistics for smaller sample sizes. MIX and MMD consistently produced least or comparable root mean square error estimates. MIX seemed to hold an edge over MMD. Both of these estimators were remarkably stable at smaller sample sizes.

CONCLUSIONS

MIX and MMD were found to be clearly superior to other methods in terms of resistance and efficiency of performance as quantified in their *RMSE* and *BIAS*. The method advocated by U.S. Water Resources Council (MMI1) performed poorly. It seems that its continued recommendation for U.S. Agencies is unwarranted. MLE and ENT typically required two orders of magnitude higher CPU time than other methods and faired poorly in performance. Based on the investigations of this study, MIX holds an edge over MMD in performance. However, the results are close for the two methods.

The results of this study indicate that, when considering the fitting of the LP3 model, MIX or MMD should be used as the methods of estimation, as they are clearly superior to the method advocated by the U.S. Water Resources Council for obtaining higher return period quantiles. They also indicate that future studies comparing several quantile estimators warrant inclusion of MMD and MIX for LP3 model.

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APPENDIX — METHOD OF MIXED MOMENTS (MIX)

MIX conserves the sample mean and variance of the untransformed data, and the sample mean of the log-transformed data. Let $\bar{x}$, s_x^2, $\bar{y}$ be the sample estimates of mean and variance of the untransformed data, and mean of the log-transformed data, respectively. Therefore, using eqns. (3), (7), and (8) with population statistics replaced by the sample estimates, the MIX parameter estimation equations are:

$$\bar{y} = c + ab \qquad \text{(A1)}$$

$$\bar{x} = \frac{\exp(c)}{(1 - a)^b} \tag{A2}$$

$$s_x^2 = \exp(2c)\left[\frac{1}{(1 - 2a)^b} - \frac{1}{(1 - a)^{2b}}\right] \tag{A3}$$

Eliminating c by combining eqns. (A1) and (A2):

$$\bar{y} - \ln \bar{x} = b[a + \ln(1 - a)] \tag{A4}$$

Again, eqns. (A2) and (A3) can be combined and c eliminated:

$$\ln\left(\frac{s_x^2 + \bar{x}^2}{\bar{x}^2}\right) = 2b \ln(1 - a) - b \ln(1 - 2a) \tag{A5}$$

Combining eqns. (A4) and (A5):

$$\frac{2 \ln(1 - a) - \ln(1 - 2a)}{\ln(1 - a) + a} = P \tag{A6}$$

where $P = \ln[(s_x^2 + \bar{x}^2)/\bar{x}^2]/(\bar{y} - \ln \bar{x})$ can be found from sample estimated $\bar{x}$, $\bar{y}$, s_x^2.

The left-hand side of eqn. (A6) depends only on parameter a and is defined for $a < 1/2$. It can be easily shown from eqn. (A6) that P approaches $-\infty$, -2, and 0 as parameter a approaches 1/2, 0 and $-\infty$ respectively. P is a smooth function over the domain $a < 1/2$.

An a–P table (reported as Table 8) was generated for a large number of values of parameter $a < 1/2$ by making use of eqn. (A6). Care was taken to perform these calculations in double precision accuracy. Thus the MIX parameter estimation procedure can be used as follows: (1) Calculate the statistic $P = \ln(1 + cv_x^2)/(\bar{y} - \ln \bar{x})$ where cv_x is the sample estimated coefficient of variation, $s_x/\bar{x}$; (2) Interpolate the value of a (Table 8) for P found above; and (3) Calculate b and c from eqns. (A4) and (A1) respectively.

For most engineering problems, the estimate of a obtained from interpolation should be sufficiently accurate. If need be, it can be further refined by the Newton–Raphson method applied to eqn. (A6) as follows:

$$f(x) = \ln\left(\frac{(1 - a)^2}{(1 - 2a)}\right) - P\ln(1 - a) - aP = 0 \tag{A7}$$

and:

$$f'(x) = \frac{-2}{(1 - a)} + \frac{2}{(1 - 2a)} + \frac{P}{(1 - a)} - P \tag{A8}$$

Therefore:

$$a_{(n+1)} = a_{(n)} - \frac{f(a_{(n)})}{f'(a_{(n)})} \tag{A9}$$

with $a_{(0)}$ being the interpolated value from Table 8.

Rao (1983) has reported some problems with convergence, if a good starting solution of a is not used in his iterative scheme. The technique of choosing the starting solution as the interpolated value, as outlined above, has resolved the aforementioned problems and always leads to fast convergence to any desired degree of accuracy.

TABLE 8

Estimating scale parameter *a* method of mixed moments (MIX)

a	*P*	*a*	*P*	*a*	*P*
−4000.00000	−0.00190	0.00000	−2.00000	0.20367	−2.81039
−3000.00000	−0.00244	0.00001	−2.00002	0.22173	−2.92392
−2000.00000	−0.00247	0.00005	−2.00013	0.23753	−3.03270
−1000.00000	−0.00626	0.00010	−2.00027	0.25575	−3.17098
−500.00000	−0.01119	0.00020	−2.00053	0.26247	−3.22584
−250.00000	−0.01977	0.00030	−2.00080	0.27701	−3.35293
−200.00000	−0.02369	0.00040	−2.00107	0.28490	−3.42712
−125.00000	−0.03451	0.00050	−2.00133	0.29326	−3.51013
−100.00000	−0.04117	0.00060	−2.00160	0.30211	−3.60366
−50.00000	−0.07052	0.00070	−2.00187	0.31153	−3.70990
−25.00000	−0.11887	0.00080	−2.00214	0.32154	−3.83172
−12.50000	−0.19675	0.00090	−2.00240	0.33000	−3.94245
−10.00000	−0.23037	0.00100	−2.00267	0.34000	−4.08398
−5.00000	−0.36956	0.00125	−2.00334	0.35000	−4.23860
−2.50000	−0.57228	0.00167	−2.00445	0.36000	−4.40844
−1.25000	−0.84064	0.00200	−2.00535	0.37000	−4.59608
−1.00000	−0.93752	0.00250	−2.00669	0.38000	−4.80482
−0.50000	−1.24592	0.00333	−2.00893	0.39000	−5.03886
−0.40000	−1.34048	0.00400	−2.01073	0.40000	−5.30371
−0.25000	−1.52001	0.00500	−2.01344	0.40500	−5.44989
−0.20000	−1.59352	0.00800	−2.02161	0.4[illegible]00	−5.60671
−0.10000	−1.76954	0.01000	−2.02709	0.41500	−5.77555
−0.05000	−1.87641	0.01250	−2.03401	0.42000	−5.95802
−0.04000	1.89966	0.01389	−2.03787	0.42500	−6.15608
−0.02500	−1.93587	0.02000	−2.05508	0.43000	−6.37210
−0.02000	−1.94830	0.02500	−2.06942	0.43500	−6.60903
−0.01250	−1.96731	0.02941	−2.08227	0.44000	−6.87052
−0.01000	−1.97375	0.03030	−2.08489	0.44500	−7.16128
−0.00800	−1.97893	0.03333	−2.09386	0.45000	−7.48738
−0.00500	−1.98677	0.04000	−2.11391	0.45500	−7.85688
−0.00400	−1.98940	0.05000	−2.14485	0.46000	−8.28086
−0.00250	−1.99336	0.06667	−2.19890	0.46500	−8.77498
−0.00200	−1.99468	0.07692	−2.23381	0.47000	−9.36239
−0.00125	−1.99667	0.08333	−2.25631	0.47500	−10.07940
−0.00100	−1.99734	0.10000	−2.31741	0.48000	−10.98753
−0.00090	−1.99760	0.11111	−2.36039	0.48500	−12.20337
−0.00080	−1.99787	0.12500	−2.41687	0.49000	−13.99187
−0.00070	−1.99813	0.13889	−2.47668	0.49500	−17.20950
−0.00060	−1.99840	0.14104	−2.48628	0.49600	−18.27875
−0.00050	−1.99867	0.14306	−2.49535	0.49700	−19.67587
−0.00040	−1.99893	0.14409	−2.50001	0.49800	−21.67429
−0.00030	−1.99920	0.14451	−2.50190	0.49900	−25.14880
−0.00020	−1.99946	0.14459	−2.50228	0.49910	−25.68150
−0.00010	−1.99973	0.14684	−2.51255	0.49920	−26.27815
−0.00005	−1.99987	0.15601	−2.55545	0.49930	−26.95593
−0.00003	−1.99992	0.16367	−2.59268	0.49940	−27.73999
−0.00001	−1.99997	0.18484	−2.70271	0.49950	−28.66924

$P = [2\ln(1 - a) - \ln(1 - 2a)]/[\ln(1 - a) + a]$

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