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EFFICIENT TESTS FOR NORMALITY, HOMOSCEDASTICITY AND SERIAL INDEPENDENCE OF REGRESSION RESIDUALS

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We use the Lagrange multiplier procedure to derive efficient joint tests for residual normality, homoscedasticity and serial independence. The tests are simple to compute and asymptotically distributed as χ^2.

1. Introduction

'Classical regression analysis' assumes the normality (N), homoscedasticity (H) and serial independence (I) of regression residuals. Violation of the *normality* assumption may lead the investigator to inaccurate inferential statements. Recently, tests for normality have been derived for the case of homoscedastic serially independent (HI) residuals [e.g., White and Macdonald (1980)]. Similarly, the effects of violation of the *homoscedasticity* assumption have been studied and tests for this have been derived for the case of normally distributed serially independent (NI) residuals [e.g., Breusch and Pagan (1979)]. Additionally, the consequences of violation of the assumption of *serial independence* have been analyzed and tests for this have been derived for the case on normally distributed homoscedastic (NH) residuals [e.g., Durbin and Watson (1950) and Breusch (1978)]. The appropriateness of these and other 'one-directional tests' (i.e., tests for either N or H or I) may depend strongly on the validity of the conditions under which these were derived (e.g., we have found that the power of our normality test — see section 2 — may be seriously affected by the presence of serial correlation). In general, it is thought that these conditions should be tested rather than assuming their validity from the start. In this paper we suggest an

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efficient 'three-directional test' for residual normality, homoscedasticity and serial independence (*NHI*) and comment on the 'one and two-directional tests' that arise as particular cases of the procedure used.

2. The test statistic

We consider the regression model given by $y_t = x_t'\beta + u_t$ with stationary residuals $u_t = \gamma_1 u_{t-1} + \ldots + \gamma_p u_{t-p} + \epsilon_t$, where x_t' is a 1 by K vector of fixed regressors including a constant satisfying the conditions set out in Amemiya (1977), β and $\gamma = (\gamma_1,\ldots,\gamma_p)$ are vectors of unknown parameters and $\epsilon_1,\ldots,\epsilon_T$ are T serially independent residuals with zero population mean. We assume that the density of ϵ_t, say $g(\epsilon_t)$, has a single mode and 'smooth contact with the ϵ-axis at the extremities'. More specifically, we assume $g(\epsilon_t)$ to be a member of the Pearson Family of distributions. This is not very restrictive due to the wide range of distributions encompassed in it (e.g., particular members of this are the Normal, Beta, Gamma, t, F and Pareto distributions). [1] This means that we can write [see Kendall and Stuart (1969, p.148)]

$$g(\epsilon_t) = \exp[\psi(\epsilon_t)] \Big/ \int_{-\infty}^{\infty} \exp[\psi(\epsilon_t)]\mathrm{d}\epsilon_t, \qquad -\infty < \epsilon_t < \infty, \tag{1}$$

where $\psi(\epsilon_t) = \int [(c_{1t} - \epsilon_t)/(c_{0t} - c_{1t}\epsilon_t + c_{2t}\epsilon_t^2)]\mathrm{d}\epsilon_t$ and $\epsilon_t = u_t - \gamma_1 u_{t-1} - \ldots - \gamma_p u_{t-p}$. Since c_{1t} denotes the mode of the density we can assume this to be the same for all t, i.e., we can set $c_{1t} = c_1$. We can show that $E[\epsilon_t^2] = c_{0t}/(1 - 3c_{2t})$ and that when $c_1 = c_{2t} = 0$, $g(\epsilon_t)$ reduces to a normal density with mean zero and variance c_{0t}. Therefore we can reparameterize our model so that $c_{2t} = c_2$ and, considering additive heteroscedasticity, set $c_{0t} = \sigma^2 + z_t'\alpha$, where z_t' is a 1 by q vector of fixed variables satisfying the conditions set out in Amemiya (1977). We now define $\phi(\epsilon_t) = \int (c_1 - \epsilon_t)/(\sigma^2 + z_t'\alpha - c_1\epsilon_t + c_2\epsilon_t^2)\mathrm{d}\epsilon_t$. Initially we proceed as if the u_t were observed. We assume to have $T+p$ observations and regard the first p residuals $u_{-p+1},\ldots,u_0$ as constants. Then we have that the logarithm of the likelihood function is given by

$$l(c_1,c_2,\sigma^2,\alpha,\gamma) = -\sum_{t=1}^{T} \ln\left[\int_{-\infty}^{\infty} \exp\phi(\epsilon_t)\mathrm{d}\epsilon_t\right] + \sum_{t=1}^{T} \phi(\epsilon_t). \tag{2}$$

[1] Our test is derived for the Pearson Family and hence, for this, we know it will have optimal properties. However, the test is not restricted to members of this family.

To derive our test statistic we make use of the Lagrange multiplier procedure. This establishes [e.g., Breusch and Pagan (1979)] that under $H_0: \theta_2 = 0$, the statistic $LM = \hat{d}_2'[\hat{\mathcal{I}}_{22} - \hat{\mathcal{I}}_{21}\hat{\mathcal{I}}_{11}^{-1}\hat{\mathcal{I}}_{12}]^{-1}\hat{d}_2$ is asymptotically distributed as a χ^2 with m degrees of freedom (χ^2_m), where $\theta = (\theta_1', \theta_2')'$ is the $m+n$ by 1 vector of all parameters in our model and θ_2 is of dimension m by 1; $d_j = \partial l(\theta)/\mathrm{d}\theta_j$ for $j = 1,2,; \mathcal{I}_{ij} = E[-\partial l(\theta)/\partial\theta_i\partial\theta_j']$ for $i,j = 1,2; l(\theta)$ is the logarithm of the likelihood function and the ^ denotes that all quantities are evaluated at the restricted maximum likelihood estimator of θ.

Here we let $\theta_1 = \sigma^2$ and $\theta_2' = (c_1, c_2, \alpha', \gamma')$. Recall that when $c_1 = c_2 = 0$, $g(\epsilon_t)$ becomes the normal density and note that when $\alpha = 0$ and $\gamma = 0$ we obtain HI residuals u_t. Hence we define our null hypothesis as $H_0: \theta_2 = 0$. Then, using the procedure described above, we can show that—after some computations—one arrives at [2]

$$\begin{aligned} LM = {} & T\left[\mu_3^2/(6\mu_2^3) + (1/24)(\mu_4/\mu_2^2 - 3)^2\right] \\ & + T\left[3\mu_1^2/(2\mu_2) - \mu_3\mu_1/\mu_2^2\right] \\ & + (1/2)\left[f'Z(Z'MZ)^{-1}Z'f\right] + T[r'r], \end{aligned} \tag{3}$$

where $\mu_j = \Sigma_{t=1}^T u_t^j/T$, $f' = (f_1, \dots, f_T)$ with $f_t = (u_t^2/\mu_2) - 1$, $Z = (z_1, \dots, z_T)'$, $M = I_T - 1(1'1)^{-1}1'$ with 1 being a T by 1 vector of ones, and $r' = (r_1, \dots, r_p)$ with $r_j = \Sigma_{t=1}^T u_t u_{t-j}/\Sigma_{t=1}^T u_t^2$.

The above statistic could be computed if the u_t were actually observed. In regression analysis, however, one does not observe the residuals. Nonetheless, it can be shown that we can replace, in (3), u_t by the Ordinary Least Squares (OLS) residuals $\hat{u}_t$ and that this does not affect the asymptotic properties of the test. We shall denote the resulting statistic by LM_{NHI}. Noting that $\Sigma_{t=1}^T \hat{u}_t = 0$ we obtain

$$\begin{aligned} LM_{NHI} = {} & T\left[b_1/6 + (b_2 - 3)^2/24\right] \\ & + (1/2)\left[\hat{f}'Z(Z'MZ)^{-1}Z'\hat{f}\right] + T[\hat{r}'\hat{r}], \end{aligned} \tag{4}$$

where $\sqrt{b_1} = \hat{\mu}_3/\hat{\mu}_2^{3/2}$, $b_2 = \hat{\mu}_4/\hat{\mu}_2^2$, $\hat{\mu}_j = \Sigma_{t=1}^T \hat{u}_t^j/T$, $\hat{f}' = (\hat{f}_1, \dots, \hat{f}_T)$ with $\hat{f}_t = (\hat{u}_t^2/\hat{\mu}_2) - 1$, and $\hat{r}' = (\hat{r}_1, \dots, \hat{r}_p)$ with $\hat{r}_j = \Sigma_{t=1}^T \hat{u}_t \hat{u}_{t-j}/\Sigma_{t=1}^T \hat{u}_t^2$. The

[2] The proof of this and related results is available from the authors.

first term in (4) is identical to the LM residual normality test for the case of HI residuals [e.g., Jarque and Bera (1980)], say LM_N. The second term is the LM homoscedasticity test for the case NI residuals [e.g., Breusch and Pagan (1979)], say LM_H. Finally, the third term is the LM serial correlation test for the case of NH residuals [e.g., Breusch (1978)], say LM_I. Therefore, it is interesting to note that these 'one-directional tests' may be combined as above, to obtain a joint test for residual NHI.

LM_{NHI} is, under H_0, asymptotically distributed as χ^2_{2+q+p} and H_0 would be refiected if the computed value of LM_{NHI} exceeded the desired significance point on the χ^2_{2+q+p} distribution. The test is asymptotically equivalent to the likelihood ratio test and this implies that it has, for large-samples, maximum local power. In addition, it is simple to compute requiring only OLS residuals. As a result, it should prove to be a useful diagnostic check in the analysis of regression residuals.

3. Concluding remarks

To finalize let us note that we can use the procedure presented in section 2 to derive efficient 'two-directional tests'. *Firstly*, we can assume N and obtain a test for HI. For this we set $c_1 = c_2 = 0$ in (2) and test $H_0' : \alpha = 0$ and $\gamma = 0$. This gives $LM_{HI} = LM_H + LM_I$ which is, under H_0'', asymptotically distributed as χ^2_{q+p}. *Secondly*, we can assume I and obtain a test for NH (this may be particularly useful in cross-sectional studies). Here we set $\gamma = 0$ in (2) and test H_0'': $c_1 = c_2 = 0$ and $\alpha = 0$. The result is $LM_{NH} = LM_N + LM_H$ which is, under H_0'', asymptotically distributed as χ^2_{2+q}. *Thirdly*, we can assume H and obtain a test for NI. For this we set $\alpha = 0$ in (2) and test $H_0''' : c_1 = c_2 = 0$ and $\gamma = 0$. This gives $LM_{NI} = LM_N + LM_I$ which is, under H_0''', asymptotically distributed as χ^2_{2+p}.

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