LOG PEARSON TYPE-3 DISTRIBUTION: PARAMETER ESTIMATION

HUYNH NGOC PHIEN and MUHAMMAD ASHRAF HIRA
Asian Institute of Technology, Bangkok (Thailand)
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ABSTRACT


Four additional methods for estimating the parameters of the log Pearson type-3 distribution are provided and appropriate computational schemes are suggested for them as well as for the method of maximum likelihood and method of (direct and indirect) moments. Simulation results indicate that the method based on the first two moments of the original data and the variance of the log-transformed values provides the best estimates, which are slightly better than those obtained from the method of maximum likelihood.

1. INTRODUCTION

The log Pearson type-3 (LP3) distribution was recommended by the Water Resources Council (W.R.C., 1967) of the U.S.A. as a base method for flood frequency analysis. The W.R.C. also recommended that its parameters should be estimated using the method of moments applied to the logarithms of observed data. However, Bobée (1975) in his careful investigation of this distribution, proposed that the method of moments be applied to the original data instead of their logarithms. Besides these two approaches based upon the moments, Condie (1977) derived a system of equations for estimating the parameters by the method of maximum likelihood. From fitting 37 long-term unregulated flood data sets in Canada, he concluded that the maximum likelihood method is superior to the method of moments. However, also based upon the standard error of the T-yr. flood Nozdryn-Plotnicki and Watt (1979), in their simulation study, found that the above methods are almost the same. They suggested the use of the method of moments recommended by the W.R.C. because of its computation ease.

Noticing the method of moments can be applied to the original data as well as to their logarithms, the present study provides additional alternatives for parameter estimation. Moreover, an attempt was also made to evaluate the performance of these methods along with the existing ones using the simulated relative errors.
2. ESTIMATION METHODS

The density function of the LP3 distribution is written as:
\[ f(x) = \left(\frac{\ln x - c}{a}\right)^{b-1} \exp\left(-\frac{\ln x - c}{a}\right) / \Gamma(b) \]
where \( a, b \) and \( c \) are the scale, shape and location parameters, respectively; and \( \Gamma(\cdot) \) is the gamma function. Let \( y = \ln x \), then the density function of \( y \) is:
\[ g(y) = \left(\frac{y - c}{a}\right)^{b-1} \exp\left(-\frac{y - c}{a}\right) / a \Gamma(b) \]
If \( a > 0 \), \( y \) has a positive skewness. In this case, \( c \leq y < +\infty \) and obviously, \( \exp(c) \leq x < +\infty \). If \( a < 0 \), \( y \) has a negative skewness. Then \( -\infty < y < c \) and \( 0 < x < \exp(c) \).

The mean, variance and skewness coefficient of \( y \) can be easily obtained as:
\[ m_1'(o) = c + ab; \quad m_2(o) = a^2b \quad (1), (2) \]
and
\[ g(o) = 2a / (|a| b^{1/2}) \quad (3) \]
where the symbol (o) is used to indicate population values. The moment of order \( k \) of \( x \) is obtained as:
\[ M_k'(o) = e^{kc(1-ka)^{-b}}, \quad 1-ka > 0, \quad k = 1, 2, \ldots \quad (4) \]
It follows from this equation that the mean, variance and skewness coefficient of \( x \) are obtained as follows:
\[ M_1'(o) = Be^c; \quad M_2(o) = Ae^{2c} \quad (5), (6) \]
\[ G(o) = C/A^{3/2} \quad (7) \]
where
\[ A = (1-2a)^{-b} - (1-a)^{-2b}, \quad B = (1-a)^{-b} \quad (8), (9) \]
and
\[ C = (1-3a)^{-b} - 3[(1-a)(1-2a)]^{-b} + 2(1-a)^{-3b} \quad (10) \]
[Note that in the expression for \( C \) given by Hoshi and Burges (1981), \( (1-a) \) in the bracket was missing.]

In the following, existing methods for parameter estimation are first presented along with suitable computational schemes. Additional alternatives are then described.

2.1. Method of W.R.C.

This is the method of moments applied to the logarithms of flood flows (referred to as the indirect method). When a sample of size \( N \) is available, the sample mean and variance are computed by the following equations:
\[ m'_1 = \bar{y} = N^{-1} \sum y_i, \quad m_2 = (N-1)^{-1} \sum (y_i - \bar{y})^2 \]

where \( y_i = \ln x_i \), \( x_i \) being an observed flow. For the skewness coefficient, the correction of bias suggested by Bobée and Robitaille (1975) should be used. In the case when one sample is available, the sample skewness coefficient is computed as follows:

\[
g = Cs \left[ 1 + 6.51/N + 20.2/N^2 + (1.48/N + 6.77/N^2)Cs^2 \right] \quad (11)
\]

where

\[
Cs = (1/N) \sum (y_i - \bar{y})^3 / \left[ (1/N) \sum (y_i - \bar{y})^2 \right]^{3/2} \quad (12)
\]

By equating the sample and population values, one obtains:

\[
b = 4/g^2; \quad a^2 = m_2 g^2 / 4 \quad (13), (14)
\]

and

\[
c = \bar{y} - ab \quad (15)
\]

In eq. 14, \( a \) is chosen to have the same sign as \( g \).

### 2.2. Method of Bobée

Bobée (1975) proposed that the method of moments be applied directly to the observed data (the direct method). By using the first three moments, \( M'_k (o) = M'_k \) where \( M'_k \) is computed from a sample, \( k = 1, 2, 3 \), he obtained the following equation:

\[
\ln \left[ \frac{(1-a)^3}{(1-3a)} \right] \ln M'_3 - 3 \ln M'_1 = \ln \left[ \frac{(1-a)^2}{(1-2a)} \right] \ln M'_2 - 2 \ln M'_1 \quad (16)
\]

When \( a \) has been determined from eq. 16, one can obtain \( b \) and \( c \) as follows:

\[
b = \ln \left( M'_2 / M'_1 \right)^2 / \ln \left( (1-a)^2 / (1-2a) \right) \quad (17)
\]

\[
c = \ln M'_1 + b \ln (1-a) \quad (18)
\]

To solve eq. 16 for \( a \), the following approximation (Kite, 1977) may be used. Let:

\[
D = \left( \ln M'_3 - 3 \ln M'_1 \right) / \left( \ln M'_2 - 2 \ln M'_1 \right); \quad E = 1/a - 3 \quad (19), (20)
\]

and

\[
F = 1/(D - 3) \quad (21)
\]

then, if \( 3.0 < D \leq 3.5 \):

\[
E = -0.47157 + 1.99955F \quad (22a)
\]

and, if \( 3.5 < D \leq 6.0 \):

\[
E = -0.23019 + 1.65262F + 0.21911F^2 - 0.04557F^3 \quad (22b)
\]
Since both $D$ and $F$ are computed from the given sample, using the equation:

$$M'_k = N^{-1} \sum X_i^k, \quad k = 1, 2, 3$$

$E$ can be obtained using eq. 22 when $3.0 < D \leq 6.0$, and thus $a$ is determined as $a = 1/(E + 3)$. However, if $D$ is not in that interval, a general iterative scheme should be applied. From eq. 16, one can write:

$$\phi(a) = \ln [(1 - a)^3/(1 - 3a)] - D \ln [(1 - a)^2/(1 - 2a)] = 0 \quad (23)$$

and, hence,

$$\phi'(a) = \frac{d\phi}{da} = \frac{2a(3(1 - 2a) - D(1 - 3a))}{(1 - a)(1 - 2a)(1 - 3a)} \quad (24)$$

Thus the Newton–Raphson method is suitable:

$$a^{(h+1)} = a^{(h)} - \frac{\phi(a^{(h)})}{\phi'(a^{(h)})}$$

where $a^{(h)}$ denotes the value of $a$ obtained in the $h$th iteration. It should be noted that $a$ must be less than $\frac{1}{3}$ for the moments $M'_1$, $M'_2$ and $M'_3$ to be defined. Bobée (1974) showed that when $a$ is small, an approximate solution to eq. 23 may be obtained as:

$$a \approx (D - 3)/[1 + (5D - 14)^{1/2}] \quad (25)$$

This approximation gives a relative error of less than 1% for $|a| < \frac{1}{30}$. It may be used as a starting (initial) value in the Newton–Raphson scheme. However, it is better to start with the value of $a$ obtained from eq. 14.

2.3. Method of maximum likelihood

Let $L$ denote the likelihood function of the LP3 distribution, then:

$$\ln L = - \sum \ln x_i - N \ln \Gamma(b) - (1/a) \sum (\ln x_i - c) + (b - 1) \sum \ln [(\ln x_i - c)/a] - N \ln |a|$$

Taking the partial derivatives of $\ln L$ with respect to $a$, $b$ and $c$, and equating each of them with zero, one arrives at the following system of equations:

$$(1/a^2) \sum (\ln x_i - c) - (1/a) N b = 0 \quad (26)$$

$$- N \Psi(b) + \sum \ln [(\ln x_i - c)/a] = 0 \quad (27)$$

$$N/a - (b - 1) \sum (\ln x_i - c)^{-1} = 0 \quad (28)$$

where

$$\Psi(b) = \frac{d}{db} \ln \Gamma(b) \quad (29)$$

which is commonly referred to as the psi or digamma function. From eq. 26,
\[ a = (N^b)^{-1} \sum (\ln x_i - c) \]

or

\[ a = (\bar{y} - c)/b \]

(30)

and from eq. 28,

\[ b = \beta / (\beta - N^2) \]

(31)

where

\[ \beta = \left[ \sum (\ln x_i - c) \right] \left[ \sum (\ln x_i - c)^{-1} \right] \]

The psi function may be computed according to the proposed formula of Condie (1977):

\[ \Psi(b) = \ln (b + 2) - (\frac{1}{2})(b + 2)^{-1} - (\frac{1}{12})(b + 2)^{-2} + (\frac{1}{120})(b + 2)^{-4} \]

\[ - (\frac{1}{252})(b + 2)^{-6} - (b + 1)^{-1} - b^{-1} \]

(32)

Since \( N > 0 \), and \( \ln x_i - c \) and \( a \) must have the same sign, it follows that \( b - 1 > 0 \) or \( b > 1 \). This requirement clearly makes the maximum likelihood estimates biased when the population does have a shape parameter \( b < 1 \), as pointed out by Condie (1977).

In order to obtain the maximum likelihood estimates, one can solve eq. 27 using a trial-and-error procedure. This may start with an assumed value of \( c \), then \( b \) and \( a \) are computed from eqs. 31 and 30, respectively, and check if eq. 27 is verified. A starting value for \( c \) may be obtained from the method of W.R.C. (1967), eq. 15.

2.4. Method of mixed moments

In seeking alternatives, the present study provides more methods for parameter estimation by combined use of moments from original data and from their logarithms. Due to the fact that the third moment, when estimated from a sample, fluctuates greatly, it was decided not to involve the skewness coefficient in those methods.

2.4.1. Method based on the first two moments of \( x \) and first moment of \( y \). By setting \( M'_1(o) = M'_1 \), \( M'_2(o) = M'_2 \) and \( m'_1(o) = m'_1 \), one obtains the following equations:

\[ c - b \ln (1 - a) = \ln M'_1; \quad 2c - b \ln (1 - 2a) = \ln M'_2 \]

(33), (34)

and

\[ c + ab = m'_1 = \bar{y} \]

(35)

It follows from eqs. 33 and 34 that:
\[ b \ln \left[ \frac{(1 - 2a)}{(1 - a)^2} \right] = \ln \left( \frac{M_1'^2}{M_2'} \right) \]
or
\[ b = \ln \left( \frac{M_1'^2}{M_2'} \right) / \ln \left[ \frac{(1 - 2a)}{(1 - a)^2} \right] \]  
(36)

From eq. 35,
\[ c = \tilde{y} - ab \]  
thus once \( a \) is determined, \( b \) can be computed from eq. 36, and once \( a \) and \( b \) have been obtained, \( c \) is completely determined from eq. 15. Now, by substitution of \( b \) and \( c \) into eq. 33, one obtains:
\[ \frac{a + \ln (1 - a)}{\ln \left( \frac{(1 - a)^2}{(1 - 2a)} \right)} = \frac{\ln M_1' - \tilde{y}}{\ln \left( \frac{M_1'^2}{M_2'} \right)} \]  
(37)

To solve for \( a \), eq. 37 is rewritten in the following form:
\[ \phi(a) = a + \ln (1 - a) - \left[ \frac{(\ln M_1' - \tilde{y})}{\ln \left( \frac{M_1'^2}{M_2'} \right)} \right] \times \left[ 2 \ln (1 - a) - \ln (1 - 2a) \right] = 0 \]  
(38)

One can easily show that
\[ \phi'(a) = \frac{-a/(1 - a)}{1 + \left\{ 2/(1 - 2a) \right\} \left\{ (\ln M_1' - \tilde{y})/\ln \left( \frac{M_1'^2}{M_2'} \right) \right\} } \]  
(39)

Thus a Newton–Raphson procedure is suitable. A starting value for \( a \) may be that given by eq. 14.

2.4.2. Method based on the first two moments of \( x \) and second moment of \( y \).

By setting \( M_1'(o) = M_1', \ M_2'(o) = M_2' \) and \( m_2(o) = m_2 \), one arrives at the following equations:
\[ c - b \ln (1 - a) = \ln M_1' \]  
\[ 2c - b \ln (1 - 2a) = \ln M_2' \]  
(33), (34)

and
\[ a^2 b = m_2 \]  
(40)

From eq. 40,
\[ b = m_2/a^2 \]  
(41)

and from eq. 33,
\[ c = \ln M_1' + (m_2/a^2) \ln (1 - a) \]  
(42)

The problem remains to determine \( a \). Eliminating \( c \) from eqs. 33 and 34 gives eq. 36, and then eqs. 36 and 41 lead to the following equation containing only \( a \):
\[ \phi(a) = a^2 \ln \left( \frac{M_1'^2}{M_2'} \right) - m_2 \ln \left[ \frac{(1 - 2a)}{(1 - a)^2} \right] = 0 \]  
(43)

Differentiation with respect to \( a \), yields:
\[ \phi'(a) = 2a \ln \left( \frac{M_1'^2}{M_2'} \right) + 2m_2a/[(1 - a)(1 - 2a)] \]  
(44)
To solve for $a$, a Newton–Raphson procedure is appropriate. The value of $a$ obtained from eq. 14 can be used as a starting value for the iterative procedure involved.

### 2.4.3. Method based on the first moment of $x$ and first two moments of $y$.

In this case, $M'_1(o) = M'_1$, $m'_1(o) = \bar{y}$ and $m_2(o) = m_2$, and thus the following equations are obtained:

$$c - b \ln (1 - a) = \ln M'_1; \quad c + ab = \bar{y}$$

and

$$a^2 b = m_2$$

Clearly,

$$b = m_2/a^2 \quad \text{and} \quad c = \bar{y} - ab = \bar{y} - m_2/a$$

Under substitution of $b$ from eq. 41 and $c$ from eq. 45 into eq. 33, one obtains:

$$m_2/a - \bar{y} + (m_2/a^2) \ln (1 - a) = -\ln M'_1$$

or

$$\phi(a) = (m_2/a)[1 + (1/a) \ln (1 - a)] - \bar{y} + \ln M'_1 = 0$$

and

$$\phi'(a) = - (m_2/a^2)[(2/a) \ln (1 - a) + (2 - a)/(1 - a)]$$

Thus a Newton–Raphson procedure can be used to solve eq. 46 for $a$. Again the value of $a$ given by eq. 14 may be used to start the iterative scheme.

### 2.4.4. Method based on the first two moments of $y$ and second moment of $x$.

By setting $m'_1(o) = m'_1 = \bar{y}$, $m_2(o) = m_2$ and $M'_2(o) = M'_2$, one obtains eqs. 35, 40 and 34. Thus $b$ and $c$ are computed by eqs. 41 and 45, respectively, once $a$ has been obtained. Under substitution of $b$ and $c$ into eq. 34, one reaches:

$$-2(m_2/a - \bar{y}) - (m_2/a^2) \ln (1 - 2a) = \ln M'_2$$

or

$$\phi(a) = (m_2/a)[2 + (1/a) \ln (1 - 2a)] - 2\bar{y} + \ln M'_2 = 0$$

Differentiation of eq. 48 with respect to $a$ gives:

$$\phi'(a) = -2(m_2/a^2)[(1/a) \ln (1 - 2a) + 2(1 - a)/(1 - 2a)]$$

From eqs. 48 and 49, it is clear that a Newton–Raphson procedure can be used to solve for $a$.

### 2.4.5. Remarks.

Recently, Read (1981) has suggested the use of other estimates to replace some equations in the system (eqs. 26–29) obtained
from the method of maximum likelihood. Obviously, one would rather replace eq. 28 by some other suitable candidate, in order to avoid the bias introduced by this method. However, this replacement does not significantly reduce computational efforts, because of the presence of eq. 27. Thus both eqs. 27 and 28 should be replaced to achieve some simpler computation. Since eq. 26 (or better eq. 30) is exactly the same as eq. 1, the first two moments of the W.R.C.'s method cannot be used. Note that if the variance and skewness of \( y \) are employed, the resulting method will be exactly the same as that of the W.R.C. With the restriction that the third moment or skewness coefficient should not be used (due to high sampling error), the other alternatives will finally turn out to be among those concerned with mixed moments.

In summary, there are altogether seven different methods for estimating the parameters of the LP3 distribution. Computationally, the method of the W.R.C., denoted WRC, is simplest, and the method of maximum likelihood (MLL) is most tedious. In the latter case a trial-and-error procedure is needed. For all the remaining five methods, namely, those of Bobée (denoted MBB) and of mixed moments (denoted MM1—MM4), the systems of equations involved can be reduced to only one equation in \( a \), which can be efficiently solved by means of the Newton—Raphson procedure.

3. SIMULATION STUDY

Since there are seven differing methods for the estimation of the LP3 parameters, a simulation study was devised in order to evaluate their performance. For this purpose, a criterion is needed. For the time being, the relative errors are used in this work. According to this criterion, the method which gives the minimum errors for all the parameters is considered the best. The relative error is computed, for example for the parameter \( a \), as follows:

\[
\text{rel}(a) = \frac{(a - \hat{a})}{a}
\]

where \( a \) is the assigned value, and \( \hat{a} \) is the estimate of \( a \) according to one of the seven techniques discussed before. The LP3 distribution involves three parameters, so a moderate large sample size would be required to estimate them reasonably. Consequently, the values chosen for \( N \) are from 20 to 80 with an increment = 5 (denoted as 20(5)80). Moreover, since the computed relative errors vary from one run to another, it was decided to repeat 100 times for each \( N \). Consequently, the mean and standard deviation for the relative errors can be computed. Thus two methods perform equally well with respect to the mean; the one giving a smaller standard deviation is obviously preferred. Finally, the computer time required in each method is also recorded in order to give some feeling about the costs involved in the data processing.
The LP3 variables can be obtained by first generating Pearson type-3 (P3) variables and then taking their exponentials. To produce a gamma variable with shape parameter \( b \), the following combined algorithm (Phien and Ruksasilp, 1981) should be used: for \( b \leq 1 \), use the algorithm by Ahrens summarized by Atkinson and Pearce (1976), and for \( b > 1 \), use that of Cheng and Feast (1979). Let \( Z \) denote the resulting variable, then the P3 variable \( Y \) obtained simply by setting:

\[
Y = aZ + c
\]

Finally, the LP3 variable \( X \) is obtained as:

\[
X = \exp(aZ + c)
\]

For the results obtained in the simulation to be meaningful, the chosen values for \( a \), \( b \) and \( c \) should be expected to be encountered in actual situations. With this in mind, they were selected to fall in the range obtained from Canadian rivers, reported by Nozdryn-Plotnicki and Watt (1979). After several test runs, six sets of parameters' values were chosen as shown in Table I, and in all computational schemes the same accuracy was employed.

From computer outputs, it was found that the relative errors are very high, and the highest error was observed in the estimation of \( b \). Except for the last method of mixed moment (MM4), the relative error was lowest in the estimation of \( a \) in most cases. Since all parameters are related to one another in their estimation, it is appropriate to rank the methods of estimation by using the average relative error of all the three parameters rather than the values corresponding to each of them. It was then observed that the rank of each method does not change very much with the sample size \( N \). Typical results are collected in Table II for \( N = 20, 40, 60 \) and \( 80 \). These results also show that the performance of each method changes only slightly with regard to the values assigned to the parameters. Therefore, the simulation results may be conveniently summarized as shown in Table III, where the frequencies of the ranks of each method are computed. Clearly, the method of the W.R.C., having only rank 5 or 6, should not be used, regardless of the fact that it is simple and is the least time-consuming method. The last

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**TABLE I**

Values of parameters employed

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<th>( c )</th>
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TABLE II
Rank of estimation methods

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I—VI = six parameters sets as numbered in Table I.
*Difficulty may be encountered in convergence of computational scheme.

TABLE III
Summarized results of ranking estimation methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Frequency with respect to rank</th>
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<tbody>
<tr>
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<td>1</td>
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<tr>
<td>WRC</td>
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<tr>
<td>MBB</td>
<td>0</td>
</tr>
<tr>
<td>MLL</td>
<td>12</td>
</tr>
<tr>
<td>MM1</td>
<td>13</td>
</tr>
<tr>
<td>MM2</td>
<td>53</td>
</tr>
<tr>
<td>MM3</td>
<td>0</td>
</tr>
<tr>
<td>MM4</td>
<td>0</td>
</tr>
</tbody>
</table>
two methods, MM3 and MM4, which are based on only one moment of $x$ and two moments of $y$, do not perform well. These three cases indicate that the involvement of more moments of $y = \ln x$ should be avoided, because all the resulting methods produce unreliable estimates for the parameters of the LP3 distribution.

The method of Bobée (MBB), having a very high frequency ($\approx 72\%$) concentrating on rank 4, is the next method to produce less reliable estimates. The results obtained in this simulation study show that the MBB is clearly inferior to the maximum likelihood method (MLL), regardless of the critical comments raised by Bobée (1975, 1979). In terms of the relative errors, the method of mixed moments based on the first two moments of $x$ and the second moment of $y$, MM2, seems to produce the most reliable estimates.

Among the seven methods under consideration, WRC consumes the least computer time. Next to the WRC is the MBB. For the three methods which produce more reliable estimates, the average computer time obtained from those corresponding to individual sets of parameters values is shown in Table IV. It is seen that the method of mixed moments based on the first two moments of $x$ and the first moment of $y$, MN1, consumes the least computer time for any sample size. Except for the case $N = 30$, the time required increases with the sample size. This is obvious because a larger value of $N$ implies more computations. Apparently, the time required differs significantly among the three methods, but none of them should cause any concern.

4. SUMMARY AND CONCLUSION

The present paper introduced four additional methods for the estimation of the LP3 parameters, using the moments of both $x$ and $y = \ln x$ with order $\leq 2$, besides those recommended by the Water Resources Council (W.R.C.), and Bobée, and the maximum likelihood. An attempt was also
made to provide a suitable computational scheme for each of the above methods. A simulation study was then carried out using six sets of values for the parameters, selected to be in the range of those obtained from the Canadian rivers. In the simulation, for any value of \( N = 20(5)80, N \) variables from the LP3 distribution were generated and 100 replications were used for each \( N \). With this relatively large number of replications, the mean and standard deviation of relative errors committed in estimating the three parameters corresponding to each method were computed. The results obtained in the simulation indicate that when three or two moments of the transformed data are used, the resulting methods, namely the WRC, MM3 and MM4, produce unreliable estimates for the parameters. The method of moments using the \( x \)-sequence as recommended by Bobée (1975) did not perform well. Among the remaining three methods, MM2, which is based on the first two moments of \( x \) and the variance of \( y \), is the best, but it is only slightly better than the method of maximum likelihood, which in turn is slightly better than the MM1, which is based on the first two moments of \( x \) and the mean of \( y \). Of course, the above conclusion is made on the basis of the limited experience gained through inspection of simulated results. Further attempt should be made to evaluate the performance of these methods in terms of the standard error of the \( T \)-yr. event as well.

ACKNOWLEDGMENTS

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REFERENCES


